CONJUNCTIONS, DISJUNCTIONS AND LEWISIAN SEMANTICS FOR COUNTERFACTUALS

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Abstract. Consider the reasonable axioms of subjunctive conditionals (1) if $p \mathrel{\Box\!\!\rightarrow} q_1$ and $p \mathrel{\Box\!\!\rightarrow} q_2$ at some world, then $p \mathrel{\Box\!\!\rightarrow} (q_1 \,\&\, q_2)$ at that world, and (2) if $p_1 \mathrel{\Box\!\!\rightarrow} q$ and $p_2 \mathrel{\Box\!\!\rightarrow} q$ at some world, then $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} q$ at that world, where $p \mathrel{\Box\!\!\rightarrow} q$ is the subjunctive conditional. I show that a Lewis-style semantics for subjunctive conditionals satisfies these axioms if and only if one makes a certain technical assumption about the closeness relation, an assumption that is probably false. I will then show how Lewisian semantics can be modified so as to assure (1) and (2) even when the technical assumption fails, and in fact in one sense the semantics actually becomes simpler then. I also give necessary and sufficient conditions on the closeness relation for the validity of the subjunctive conditional law of excluded middle $(p \mathrel{\Box\!\!\rightarrow} q) \vee (p \mathrel{\Box\!\!\rightarrow} \sim q)$.

1. Introduction

Write $p \mathrel{\Box\!\!\rightarrow} q$ for the subjunctive conditional that were p to hold, q would hold. On David Lewis's semantics for subjunctive conditionals, $p \mathrel{\Box\!\!\rightarrow} q$ holds if and only if either

(a) There is no possible world at which p holds, or

(b) There is a possible world w_1 at which both p and q hold and which is such that any world w_2 at which both p and $\sim q$ hold is further from the actual world than w_1 is.

Now, the following two potential axioms of subjunctive logic:

(1) $$(p \mathrel{\Box\!\!\rightarrow} q_1) \,\&(p \mathrel{\Box\!\!\rightarrow} q_2) \text{ at } w \supset (p \mathrel{\Box\!\!\rightarrow} (q_1 \,\&\, q_2)) \text{ at } w$$

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and

(2) $$(p_1 \mathbin{\Box\!\!\to} q) \,\&(p_2 \mathbin{\Box\!\!\to} q) \text{ at } w \supset ((p_1 \vee p_2) \mathbin{\Box\!\!\to} q) \text{ at } w,$$

where $\supset$ is material implication, are very plausible. For, the subjunctive conditional says what would happen were some condition realized. If under that condition q_1 would hold and under that condition q_2 would hold, then surely under that condition $q_1 \,\&\, q_2$ would also hold. Likewise, if some proposition would hold under one condition and would also hold under another condition, then it is hard to deny that it would hold under the disjunction of these two conditions. Axioms (1) and (2) simply say that this intuition is correct at w.

One way to argue for the intuition is to use the Lewisian duality between the *would* conditional $\Box\!\!\to$ and the *might* conditional $\Diamond\!\!\to$ according to which duality $p \mathbin{\Diamond\!\!\to} q$ holds if and only if $\sim(p \mathbin{\Box\!\!\to} \sim q)$ does: if q might hold were p to hold, then it is not the case that were p to hold, $\sim q$ would hold, and conversely. According to this duality, axiom (1) says that if $P \mathbin{\Diamond\!\!\to} (Q_1 \vee Q_2)$ at w, then we cannot have both $P \mathbin{\Box\!\!\to} \sim Q_1$ and $P \mathbin{\Box\!\!\to} \sim Q_2$ at w. And, after all, how could it be the case that under some condition neither disjunct would happen, but with it still being the case that under that condition the disjunction might happen? A dual formulation of (2) says that if $(P_1 \vee P_2) \mathbin{\Diamond\!\!\to} Q$ at w, then we do not have both $P_1 \mathbin{\Box\!\!\to} Q$ and $P_2 \mathbin{\Box\!\!\to} Q$ at w, which is also plausible. I will add that as far as my intuitions go, (1) is more plausible than (2), though we shall see that under some plausible background assumptions, both axioms are equivalent.

I will derive necessary and sufficient conditions on the closeness relation for (1) and (2) to hold and argue that most likely these conditions are not satisfied. Given the high plausibility of these axioms, we should reject Lewisian semantics. Fortunately, I will show that there is an attractive modification of Lewisian semantics that verifies these two axioms.

More precisely, let a *frame* be a quadruple $\langle W, P, \top, < \rangle$, where W ("worlds") and P ("propositions") are sets, $\top$ ("truth-at") is a relation between members of P and members of W, and for every member w of W, $<_w$ ("closeness") is a relation between members of W, where the following natural axioms hold:

(c) there is a map A from the powerset of W to P such that if $S \subseteq W$, then $w \top (AS)$ holds if and only if $w \in S$;

(d) there are maps $\& : P \times P \to P$, $\vee : P \times P \to P$, and $\sim : P \to P$ such that $T_{p \,\&\, q} = T_p \cap T_q$, $T_{p \vee q} = T_p \cup T_q$ and $T_{\sim p} = W \backslash T_p$, where $T_r = \{w \in W : p \top w\}$;

(e) $<_w$ is transitive, i.e., $\alpha <_w \beta$ and $\beta <_w \gamma$ imply $\alpha <_w \gamma$;

(f) $<_w$ is asymmetric, i.e., if $\alpha <_w \beta$ then not $\beta <_w \alpha$; and

(g) $<_w$ has the property that $w <_w w'$ for every $w' \neq w$,

By abuse of notation I will write Ax for $A\{x\}$.

The intended interpretation is that W is the worlds, P is the propositions, $\top$ is the truth-at relation that holds between a proposition and a world provided the proposition is true at that world, and $w_1 <_w w_2$ holds provided that w_1 is closer to w than w_2 is. Because of this, I will at times say that $p \in P$ is "possible" provided there is a $w \in W$ such that $p \top w$, and will say in such a case that p "holds at w". Then, (c) says that for any countable set of worlds there is a proposition asserting that some world from that set is actual, (d) asserts that one can conjoin, disjoin and negate propositions, while (e)–(f) are extremely reasonable conditions we would want $<_w$ to satisfy if were it to have any chance of counting as the closeness relation. Note that in fact (d) follows from (c).

If one wishes to follow Lewis in assuming that propositions just *are* sets of worlds, one can let P be the powerset of W, and define $p \top w$ to hold if and only if $w \in p$.

Note that there are some technical difficulties if one thinks with Pruss (2001) that the collection of all worlds is not a set. If so, then W will not be all worlds, but some relevant set of them, and P will not be all propositions, but again some relevant set of them.

Given a frame $\langle W, P, \top, < \rangle$, we can define the Lewisian counterfactual by saying that

$$(p \mathrel{\Box\!\!\rightarrow} q) \text{ at } w$$

holds for p and q in P and w in W if and only if either there is no $w_1 \in W$ such that $p \top w_1$, or there is a $w_1 \in W$ satisfying $p \top w_1$ and $q \top w_1$ and such that $w_1 <_w w_2$ for all w_2 such that $p \top w_2$ and $q \top w_2$.

The closeness relation in the intended interpretation on Lewis's view is qualitative similarity of worlds, with a greater weight being placed on nomic similarity than on similarity of particular fact (Lewis, 1979). This makes (e)–(g) very plausible.

I will argue that on some highly plausible judgments about closeness, it is implausible that (1) and (2) hold under the intended interpretation. In fact, I will give necessary and sufficient conditions for a frame to verify axioms (1) and (2), and show that this condition is not met on some very reasonable interpretations of the closeness relation.

It is important to note that Lewis's own assumptions about the $<_w$ relations *do* imply that these necessary and sufficient conditions are met. However, this only means that Lewis's assumptions about what the $<_w$ relations are likely false. Fortunately, Lewis's semantics can be modified so that (1) and (2) hold despite these assumptions being false, as we shall eventually see.

As the above formulation shows, I am assuming S5 throughout.

2. A necessary and sufficient condition for (1) and (2)

Given a frame $\langle W, P, \top, < \rangle$, we say that "the closenesses of w_1 and w_2 to a world w are incommensurable", or more briefly that "w_1 and w_2 are w-incommensurable", and write $w_1 \sim_w w_2$, providing that *none* of the following three conditions hold: $w_1 <_w w_2$ or $w_2 <_w w_1$ or $w_1 = w_2$. It is plausible that w-incommensurability in fact happens. If w_0 is the actual world, then plausibly a world just like ours but with the moon being made of blue cheese and a world just like ours but with the moon being made of camembert are w-incommensurable. I will say that "there are incommensurable worlds" (relative to a frame) provided there are w_1, w_2 and w such that w_1 and w_2 are w-incommensurable.

It is an easy exercise for the reader to verify that if w_1 and w_2 are w-incommensurable, then it is neither true at w that $(Aw_1 \vee Aw_2) \mathrel{\Box\!\!\rightarrow} Aw_1$ nor that $(Aw_1 \vee Aw_2) \mathrel{\Box\!\!\rightarrow} \sim Aw_1$ on Lewis-style semantics. Hence, w-incommensurable worlds would provide a Lewisian counterexample to the subjunctive conditional law of excluded middle which claims that, necessarily, for any propositions p and q, either $p \mathrel{\Box\!\!\rightarrow} q$ or $p \mathrel{\Box\!\!\rightarrow} \sim q$.

The notion of incommensurability lets us characterize when (1) and (2) hold.

Theorem 1. *The following conditions on a frame are equivalent:*

(i) *Axiom* (1) *holds for all* p *and* q *in* P *and every* w *in* W*;*

(ii) *Axiom* (2) *holds for all* p *and* q *in* P *and every* w *in* W*;*

(iii) *For every quintuple of worlds* α, β, γ, δ *and* w *if* $\alpha <_w \beta$ *and* $\gamma <_w \delta$ *and* β *and* γ *are* w*-incommensurable, then* $\alpha <_w \delta$.

In particular, if there are no incommensurable worlds, (1) and (2) hold. I will call the final condition in the Theorem the "quintuple condition." It can be thought of as a transitivity-type of condition. If incommensurable pairs of worlds satisfied the same order relations so that $\alpha <_w \beta$ if and only if $\alpha' <_w \beta'$ whenever $\alpha \sim_w \alpha'$ and $\beta \sim_w \beta'$. The proof of the Theorem will be given in Section 6.

Lewis's *own* axiomatization is such as to imply the quintuple condition in our notation and hence yield (1) and (2). We shall now proceed to argue on intuitive grounds that on the intended interpretation the quintuple condition fails.

3. The quintuple condition

3.1. **Numerical closeness.** The quintuple condition would hold if there were some function c that assigned a number (or more generally an element of some totally ordered set) to every pair of worlds w_1 and w_2 in such a way that $w_1 <_w w_2$ held if and only if $c(w, w_1) < c(w, w_2)$. One thinks of calculating, say, the volume of space-time (using some non-standard version of arithmetic to handle infinities here) in which there are any difference between w and w_1 and calling it $c(w, w_1)$.

For if there were some such a numerical assignment, β and γ would be w-incommensurable if and only if $c(w, \beta) = c(w, \gamma)$. If $\alpha <_w \beta$ and $\gamma <_w \delta$, then $c(w, \alpha) < c(w, \beta)$ and $c(w, \gamma) < c(w, \delta)$ and it would follow then that $c(w, \alpha) < c(w, \delta)$ and hence $\alpha <_w \delta$. It is tempting to think of the closeness of worlds as having such a numerical (or quasi-numerical in the general totally ordered set case) measure, so that w-incommensurability can become assimilated to the claim that two worlds are equally close to w because, e.g., they are in some way indistinguishable. If that were so, then the quintuple condition would hold.

With a single numerical assignment, w-incommensurability simply means equal distance. On the multiple parameter approach, w-incommensurability means either equal distance or incomparability. I will now give two arguments for the multiple parameter approach, first one with numerical and the one with non-numerical parameters, each of which will yield a plausible counterexample to the quintuple condition, and then I shall give a third counterexample that involves worlds with an infinite number of objects.

3.2. **Multiple parameter.** It is implausible that there should be such a numerical assignment of closenessses that yields the truth of the quintuple condition. In general, we would expect a multiple-parameter way of measuring closeness. Thus, to a pair of worlds α and β one might have to assign an n-tuple of numbers, $\mathbf{c}(\alpha,\beta) = \langle c_1(\alpha,\beta),\ldots,c_n(\alpha,\beta)\rangle$, each coordinate of which measures the closeness *in some respect* between these two worlds. If the multiple parameters are not reducible to a single parameter by some operation such as addition with appropriate weights[1], then then one might reasonably say that $w_1 <_w w_2$ if and only if for all i we have $c_i(w,w_1) \le c_i(w,w_2)$ and there is some i such that $c_i(w,w_1) < c_i(w,w_2)$. If the closeness of worlds is multiply-parametric in this sense, then we cannot expect to have the quintuple condition. To see this already in the case where $n = 2$, observe that we should intuitively be easily able to find a quintuple of worlds α, β, γ, δ and w such that, say, $\mathbf{c}(w,\alpha) = \langle 1,1\rangle$, $\mathbf{c}(w,\beta) = \langle 1,2\rangle$, $\mathbf{c}(w,\gamma) = \langle 2,1/2\rangle$ and $\mathbf{c}(w,\delta) = \langle 2,3/4\rangle$. If we can find such a quintuple, then we will have

[1] I am grateful to an anonymous referee for this suggestion. The referee also suggested multiplication, but that is less promising, because it would mean that arbitrarily high degrees of closeness can be achieved simply by making two worlds be very close merely in respect of any one parameter

$\alpha <_w \beta$ and $\gamma <_w \delta$, and β and γ will be incommensurable, but we will not have $\alpha <_w \delta$.

3.3. **Mimsiness and slithiness.** But is it plausible that there is no reducibility of multiple measures of closeness to a single scale? I think so. Consider a possible world w whose denizens in addition to the kinds of properties that denizens of our world have also have two new basic physical properties I will call *mimsiness* and *slithiness*. Both of these have a numerical scale to them. Thus, some particles in w have a mimsiness of 7 mims and a slithiness of -12.4 sliths. However, these two properties do not affect any other basic properties or one another, in the sense that to predict what basic properties other than mimsiness (respectively, slithiness) will be exhibited after some experiment one does not need to know how mimsy any of the particles in the experiment were. In particular, to know how mimsy (respectively, slithy) or charged the result of some particle collision will be, we do not need to know how slithy the incoming particles were. If one has any verificationist qualms, we can suppose that the denizens of w observe mimsiness and slithiness directly. There are well defined laws of nature governing how the mimsiness of particles behaves, say, in a collision of particles and how the slithiness of particles behaves. It might be, say, that when one slithy particle hits another slithy one, slithiness is redistributed between the particles in proportion to their respective electric charges.

There are, let us suppose, mimsiness and slithiness conservation laws which hold always, and the standard conservation laws hold for all the other properties much as they do in our world. However, because of the way slithiness and mimsiness fail to affect one another and fail to affect any other basic physical properties, there are no cases where some quantity of one of the other properties, say charge, gets transformed into some quantity of slithiness or mimsiness or where some slithiness gets transformed into, say, charge. Because of this, there is no non-arbitrary way to extend the measurement system for other properties to cover slithiness and mimsiness. Slithiness may be measured in sliths and mimsiness in mims, but the physics I assume is such that there is no way of finding a relation between sliths and mims so as to measure the two on a common scale or to define any sort of energy function that includes them both. I will also assume that slithiness and mimsiness are measured on a continuous scale, so one cannot measure mimsiness with respect to some natural unit like "the smallest physically possible non-zero amount of mimsiness."

Without any physical interrelation here, any choice to combine sliths and mims into a single scale would be *ad hoc*. Sure, one could measure the overall slithiness-and-mimsiness difference between two particles via any number of formulae, such as $|s_1 - s_2| + |m_1 - m_2|$ where one stipulates that s_i is measured in sliths and m_i in mims. But without any kind of physical interrelation between the quantities, the choice of formula is arbitrary: one could, just as well, choose $|s_1 - s_2|^3 +$

$(\pi/7)\arctan|m_1 - m2|$ and there is no objective fact of the matter as to which formula is right.

Thus, we can measure how close two worlds are in terms of mimsiness and how close they are in terms of slithiness, but there is no way to interrelate these two measurements to come up with an overall closeness. Now consider some very simple worlds α, β, γ, δ and w, each of which contains exactly one particle of the same type. The particle in w has 0.1 mims and 0.1 sliths. On the other hand, the particle in α has 1 mim and 1 slith; the particle in β has 1 mim and 2 sliths; that in γ has 2 mims and 0.5 sliths; that in δ has 2 mims and 0.75 sliths. It is very natural then to say that $\alpha <_w \beta$ and $\gamma <_w \delta$, since clearly γ is more similar to w than δ is in terms of the slithiness of its particle while no less similar in terms of the mimsiness of the particle, and a corresponding thing can be said about α and β. But, very plausibly, β and γ are w-incommensurable: any choice of which one is closer to w is *ad hoc*.

World β is closer to w in terms of mimsiness and world γ in terms of slithiness. But since there is no way of counting mimsiness and slithiness on a common scale, there is no fact of the matter that β is overall closer to w or that γ is—at least if the facts about closeness are supposed to supervene on facts about the distribution and kinds of objects in the two worlds, as in Lewis's system they are. But for exactly the same reason there is no fact of the matter that α is closer to w than δ is or *vice versa*. It looks at first sight as if α would be closer to w, since α's particle has 1 mim and 1 slith, while δ's has 2 mims and 0.75 sliths, so that the total difference from w in respect of the sum total of the mimsiness/slithiness of particles in the case of α is $(1 - 0.1) + (1 - 0.1) = 1.8$ while in the case of δ it is $(2 - 0.1) + (0.75 - 0.1) = 2.55$. However, such sums make no sense, because one cannot meaningfully add mims to sliths any more than one can add kilograms to meters. Therefore, one has to say that α and δ are w-incommensurable just as β and γ are. Hence, the quintuple condition is violated.

The above argument is formulated in a way that is neutral between two claims. On one view, it is a not a fact that α is closer than δ and it is not a fact that δ is closer than α (and similarly for β and γ), but it is a fact that neither is closer than the other. On the other view, there simply is no fact of the matter as to which is closer. I prefer the first formulation, but either will do to support the argument.

A crucial question here is whether it is possible to have properties such as described above. Are they not, as one referee asked, "idle wheels" given their lack of interaction with everything else? I respond that they *may* be idle, but it does not follow that they are impossible. Consider a three-dimensional Newtonian universe with a bunch of point particles but no forces. Fix an arbitrary coordinate system. Each particle then has six properties, namely the coordinate components (x_1, x_2, x_3) of the position and the coordinate components (p_1, p_2, p_3) of the momentum. The particles do not interact with one another, and the property-pairs (x_1, p_1), (x_2, p_2) and (x_3, p_3) are such that no pair interacts with any other pair: to know the value of (x_1, p_1) for a given particle at a given time, no information about the values of (x_2, p_2) and (x_3, p_3) for any particle at any time is of any use. Hence, it seems possible to have a pair of properties that does not interact with any other pair of properties. But surely nothing hangs on the distinction between a property and a pair of properties, since we can always re-encode a pair of quantifiable properties as a single quantifiable property, given that the cardinality of the set of real numbers is equal to the cardinality of the set of pairs of real numbers. Moreover, we may assume that mimsiness comes associated with a property mimsiness*, perhaps related to it in the way that momentum along the first axis is to position along the first axis, and that slithiness comes associated with slithiness*, but that mimsiness-mimsiness* pairs have no interaction with slithiness-slithiness* pairs. In fact, depending on one's view of space, we might even take the analogy to be an identity, and define mimsiness as x_1 and slithiness as x_2.

All that said, I have no solid *positive* argument for the possibility of something like mimsiness and slithiness. But the possibility seems plausible. We can, however, do a little better, with a bit of handwaving.

3.4. **Artists and mathematicians.** Let w be the actual world. Consider a sequence of worlds slightly modified from w. Let α be a world where some world-historically unimportant Joe Shmoe was a better mathematician than and equally good artist as he is in w. Let β be a world where he is an even better mathematician than in w and just as good an artist. Let γ be a world where he is no better a mathematician than he is in the actual world but where he is a somewhat better artist. Let δ be a world where he is no better a mathematician but an even better artist than in γ. Then, $\alpha <_w \beta$ and $\delta <_w \gamma$. But unless there is some non-arbitrary single way of measuring artistry and mathematical ability, which presumably there is not, neither of β and γ is closer to the actual world, and hence they are incommensurable. But so are α and δ.

It is tempting to say that as I described things, it might be that Joe Shmoe is a better artist in δ than he is a mathematician in α, but in fact there is no way of comparing artists and chess players—they may well be incommensurable, except perhaps in some "extreme" cases. Perhaps we can say that Hitler was a worse artist than Newton a mathematician, but there is no fact of the matter that Newton was a better mathematician than Michelangelo an artist or *vice versa*. One might hypothesize, for instance, that we have a single notion of "genius", and we deem the non-genius to be worse in her special field than the genius in hers, even if the two fields of expertise are as different as painting and mathematics, but we do not have any objective way of ranking geniuses across fields of specialty, perhaps even within fields of specialty (a referee suggests that neither was Gödel was the better logician than Tarski nor *vice versa*), and we have no objective way of ranking mid-range experts across fields. Whatever the explanation, it is plausible that we cannot make claims as that Joe Shmoe is a better artist in δ than he is a mathematician in α, as long as he is close to the mid-range in both fields in both worlds.

One might object that one *can* rank people within different fields on a uniform scale by using percentile ranks. How much better x is at activity A in w_1 than y is at B in w_2 is determined by the percentile rank of x among doers of A in w_1 and the percentile rank of y among doers of B in w_2. But that cannot be right, first, because there is no way of ranking geniuses even within a single field, and, second, because excellence is not a relative term. Salieri would not have been a greater composer had Mozart simply turned his talents to long-distance running instead, even though Salieri's percentile rank would have been higher.

3.5. **The line of balls.** For the third counterexample to the quintuple principle, I will begin with the simplest formulation, and then we will add some possible complications. Let w be a Newtonian world without gravity. Rigidly fix an arbitrary unit and coordinate system. Suppose that the only physical objects the world contains are indiscernible, uniform, rigid balls of radius 0.01 The balls persist motionlessly for all times t between $-\infty$ and $+\infty$, and are centered at coordinates $(n, 0, 0)$, where n ranges over the integers. Thus, there is a line of balls, spaced one unit a part, infinite in extent in both directions. I will call the ball at $(n, 0, 0)$ "ball number n".

Now, worlds α, β, γ and δ will all be just like w except that each has only a subset of the balls. Specifically, in α, we only have all and only those balls that are at coordinates $(n, 0, 0)$ with $n \leq -5$. In β, we have all and only those balls at $(n, 0, 0)$ with $n \leq -10$. In γ, we have all and only those balls at $(n, 0, 0)$ with $n \geq 5$. Finally, in δ, we have all and only those balls that are at $(n, 0, 0)$ with $n \geq 10$.

Then, $\alpha <_w \beta$ because α and β have the same laws of nature, but α precisely coincides with w in a strictly greater area of space-time than β does. For exactly the same reasons, $\gamma <_w \delta$, as γ precisely coincides with w in a strictly greater area of space-time than β does. Recall that Lewis (1979) lists the spatio-temporal extent of exact coincidence as the second-most important criterion of closeness of worlds, after coincidence of laws, so this is not only intuitively true, but agrees with Lewis's explicit criteria.

The hardest part is now to show that β and γ are w-incommensurable and that so are α and δ. Take first the case of β and γ. Observe that the extent of spatio-temporal coincidence with w is exactly the same for both β and γ. Both β and γ have a semi-infinite line of balls. In the case of β, the line stretches in the negative direction along the x-axis. In the case of γ, it stretches in the positive direction along the x-axis. Neither line of balls is a subset of the other, as was the case between α and β or between γ and δ. In fact, β and γ have no balls in common. The set of the coordinates of the centers of the balls in β is simply the reflection in the plane $x = -5/2$ of the set of the coordinates of the balls in γ. We thus have a complete symmetry between the ways in which β is similar to w and the ways in which γ is similar to w, and we must conclude that neither can β be closer to w than γ is, nor can γ be closer to w than β is.

And of course it would be mistaken to say that γ is closer to w than β is because γ contains balls numbered $5, 6, 7, 8, 9, 10, 11, 12, \ldots$ while β contains fewer balls, having only those numbered $-10, -11, -12, \ldots$. For, after all, both sets of numbers contain exactly the same number of entries where the size of a set is measured in terms of cardinality, while if one uses containment-based considerations to measure the size of a set, then neither is larger than the other because in fact they have no elements in common.

Nor would it do to argue that γ is closer to w than β is because α is closer to w than β is, and γ is just a reflection of α about the plane $x = 0$ so γ and α must be equidistant to w. For, in exactly the same sense, γ is just a reflection of β about the plane $x = -5/2$, and so by the same reasoning they should be equidistant to w, unless there is "something special" about the plane $x = 0$. But the coordinate system is arbitrary and so there is nothing special about that plane over and beyond other planes.

Hence, β and γ are w-incommensurable. By exactly the same reasoning, α and δ are w-incommensurable: each coincides with w in respect of a semi-infinite line of balls, as well as everywhere outside the balls of w, and the two semi-infinite lines of balls have no entries in common.

Therefore, we have $\alpha <_w \beta$ and $\gamma <_w \delta$, while β and γ are w-incommensurable, and α and δ are also w-incommensurable. Hence, the quintuple condition fails, and so Lewis's semantics fails to yield (1) or (2).

Objection 1. The conclusion that $\alpha <_w \beta$ (or, *mutatis mutandis*, that $\gamma <_w \delta$) was overhasty. After all, both α and β agree with w along a semi-infinite line of balls, and everywhere outside the balls of w. Thus, they each agree with respect to precisely $\aleph_0$ balls.

However, as we already saw above, cardinality is not the only way of comparing sets. Containment considerations are also applicable, and the balls of α are a strict superset of the balls of β, while the spatio-temporal area of disagreement between α and w is a strict subset of the spatio-temporal area of disagreement between β and w.

And in fact, assuming Lewis's semantics for counterfactuals, we can give the following argument for α being closer to w than β is. Given the Newtonian laws and the arrangement above, the existence of each of the balls in w is causally independent of the existence of the other ones (if one is a theist who thinks that God necessarily exists, then one will say they all have a common cause, but that is all). Now, let w_m be a world just like w but only containing those balls at $(n, 0, 0)$ for $n \leq m$. The causal independence claim then implies that, for each m, it is true that were balls number $m+1, m+2, m+3, ...$ not to have existed, balls number $m, m-1, m-2, ...$ would still have existed. Moreover, it is highly plausible that were balls number $m+1, m+2, m+3, ...$ not to have existed, then it is precisely world w_m that would have been actual. But, given Lewis's semantics for counterfactuals, this implies that w_m is the closest world to w at which balls number $m+1, m+2, m+3, ...$ do not exist. In particular, it follows that $w_m <_w w_{m-1}$. Since m was arbitrary, it follows that in general $w_m <_w w_n$ whenever $n < m$. Since $\alpha = w_{-5}$ and $\beta = w_{-10}$, it follows that $\alpha <_w \beta$.

Objection 2. The original set-up supposed Newtonian laws to hold in all the worlds under consideration. But David Lewis thought that laws were statements that would appear in a description of the world that best balances informativeness and brevity. Newtonian laws would not appear in such a description of w. Moreover, perhaps, different laws would hold in w, on the one hand, and in α, β, γ and δ, since the optimal description of w might say that there is a doubly-infinite line of balls.

All the above is true on Lewis's controversial account of laws. However, it does not affect the argument. For while above I did indeed assume for simplicity that w, α, β, γ and δ all have the same laws, all I need is the weaker assumption that the laws of no one of α, β, γ and δ are closer to those of w than the laws of any other one of those four worlds.

Objection 3. The argument assumes that one can identify balls and positions in space-time across possible worlds even though the balls and positions are in fact indiscernible. This is unfair as part of an argument against a counterpart theorist like Lewis.

Two responses can be made. First, in a Lewisian framework one can have a context-dependent counterpart relation. Once one fixes an appropriate such relation, one with the property that counterparts of a pair of balls are the same distance apart as the original two balls, the above construction works as long as distinct indiscernibles are possible.

Second, one can simply mark up all the balls in w in some insignificant way. For instance, one might suppose that each ball in w has two red dots on it, with every ball having its dots at a different distance apart. Thus, each ball has a distinct property: its "dot-distance". The crucial thing to ensure is that pairs of worlds w_1 and w_2 that consist of disjoint half-lines of balls be w-incommensurable—this condition will ensure that once we construct α, β, γ and δ we will have the crucial w-incommensurability of α and δ as well as that of β and γ.

One might conceivably lose the incommensurability if there is some kind of lawlike pattern to the dot-distances as a function of ball number in w, a lawlike pattern that somehow ensures that, say, α is closer in law to w than δ is, if laws supervene on lawlike patterns, as Lewis thought. However, it is plausible that things can be arranged so that there is no such lawlike pattern, say by ensuring that the dot-distances as a function of ball number are distributed in such a sufficiently random way.

3.6. **What follows.** Given the above counterexamples to the quintuple condition and given Theorem 1, it follows that the highly plausible axioms (1) and (2) are false on Lewis-style semantics if one makes plausible assumptions about the results of closeness measurements. Now, Lewis himself thought that the closeness relation is going to be such as to ground (1) and (2). Lewis's own assumptions about the closeness relation (Lewis, 1973, 14) can easily be proved to be equivalent to the following list:

(h) For all $\alpha \neq w$, $w <_w \alpha$;
(i) the relation $<_w$ is transitive;
(j) w-incommensurability is transitive; and
(k) the quintuple condition holds.

By our Theorem, it does follow from these that (1) and (2) hold.

Since (k) is false under the a very plausible interpretation of the closeness relation, there is a problem here for Lewis's account of counterfactuals. Either the system makes an implausible assumption about the closeness relation, or else the system does not satisfy the highly intuitive axioms (1) and (2).

4. A partial solution

Fortunately, there is a fairly simple and natural solution. Before we give it, let us look at what the failure of (1) and (2) looks like.

A strong failure of (1) would be a case where $p \Box\!\!\rightarrow q_1$ at w and $p \Box\!\!\rightarrow q_2$ at w while $p \Box\!\!\rightarrow \sim(q_1 \,\&\, q_2)$ at w, even though p is possible. Such a thing cannot in fact occur if p is possible on our standard assumptions (e)–(g) about $<_w$, even without the quintuple condition.

Theorem 2. *Given a frame, there cannot be a strong failure of* (1) *or of* (2)*. In other words:*

(i) *If* $p \mathrel{\Box\!\!\rightarrow} q_1$ *at* w, $p \mathrel{\Box\!\!\rightarrow} q_2$ *at* w *and* $p \mathrel{\Box\!\!\rightarrow} {\sim}(q_1 \,\&\, q_2)$, *then* p *is impossible.*

(ii) *If* $p_1 \mathrel{\Box\!\!\rightarrow} q$ *at* w, $p_2 \mathrel{\Box\!\!\rightarrow} q$ *at* w *and* $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} {\sim}\, q$, *then* $p_1 \vee p_2$ *is impossible.*

Because the failure of (1) and (2) is not very severe, there is hope for an extension of Lewis's semantics for subjunctive conditionals such that if $p \mathrel{\Box\!\!\rightarrow} q$ is true on Lewis's semantics, it remains true on the modified semantics, and such that if $p \mathrel{\Box\!\!\rightarrow} {\sim}\, q$ is true on Lewis's semantics, it is not the case that $p \mathrel{\Box\!\!\rightarrow} q$ on the modified semantics. Such a semantics can in fact be given, assuming as always our conditions (c)–(g) from Section 1. Let us say, given a frame $\langle W, P, \top, < \rangle$, that $p \mathrel{\Box\!\!\rightarrow}^* q$ at w provided:

(l) For any world w_1 such that $p \top w_1$ and $({\sim}\, q) \top w_1$, there is a world w_2 such that $p \top w_2$ and $q \top w_2$, with the properties that $w_2 <_w w_1$ and there is no world w_3 such that all three of the following conditions hold: $w_3 <_w w_2$, $p \top w_3$ and $({\sim}\, q) \top w_3$.

Observe how this definition has the advantage over Lewis's in that the case where p is impossible is not being treated separately. It is simpler in the sense of not being disjunctive.

Theorem 3. *If* $p \mathrel{\Box\!\!\rightarrow} q$ *at* w*, then* $p \mathrel{\Box\!\!\rightarrow}^* q$ *at* w*, and if* $p \mathrel{\Box\!\!\rightarrow}^* q$ *at* w *and* $p \mathrel{\Box\!\!\rightarrow}^* {\sim} q$ *at* w*, then* p *is impossible, i.e., there is no* w' *such that* $p \top w'$*. Moreover,* (1) *and* (2) *hold with* $\mathrel{\Box\!\!\rightarrow}^*$ *in place of* $\mathrel{\Box\!\!\rightarrow}$*.*

The proof is given in Section 6.

5. Philosophical conclusions

Lewis's subjunctive conditional behaves poorly with respect to finite conjunctions of consequents and disjunctions of antecedents on reasonable assumptions on the interpretation of the closeness relation. We give a necessary and sufficient condition for the closeness relation to be such as to ensure good behavior. Lewis does in effect assume this condition to hold, but it is an unreasonable assumption. However, we can modify Lewis's conditional to behave well with respect to our finite conjunctions and disjunctions without making any such unreasonable assumption on the closeness relation.

6. Proofs

It suffices to prove all our Theorems in the special case where the set of propositions is the power set 2^W of W and $p \top w$ if and only if $w \in p$. To see this, given a frame $\mathcal{F} = \langle W, P, \top, < \rangle$, define the frame $\mathcal{F}^* = \left\langle W, 2^W, \ni, < \right\rangle$ where $p \ni w$ if and only $w \in p$, and given this frame let $AS = S$, $p \,\&\, q = p \cap q$, $p \vee q = p \cup q$, ${\sim} p = W \backslash p$ and similarly for infinite cases. Now observe that every single numbered statement in each of our Theorems holds for $\mathcal{F}$ if and only if it holds for $\mathcal{F}^*$—this is a simple exercise for the reader. Consequently, we need only prove the results in the case of $\mathcal{F}^*$, and so below we always assume that our propositions are subsets of W. I will generally use the "logical" symbols &, $\vee$ and $\sim$ in "propositional contexts" and $\cap$, $\cup$ and $\setminus$ in "set theoretic contexts".

Say that a world α "witnesses $p \Box\!\!\rightarrow q$ at w" provided that $\alpha \in p \cap q$ and every world $\beta \in p \setminus q$ satisfies $\alpha <_w \beta$ (the term "to witness" is Steven Kuhn's). Then, $p \Box\!\!\rightarrow q$ at w if and only if either p is the empty set or there is a world α witnessing to $p \Box\!\!\rightarrow q$ at w.

Proof of Theorem 1. First we prove that (iii) implies (i) and (ii). To do this, assume (iii) and to show (i) suppose first we have $p \Box\!\!\rightarrow q_1$ at w and $p \Box\!\!\rightarrow q_2$ at w.

Now if p is empty, it trivially follows that $p \Box\!\!\rightarrow (q_1 \cap \& q_2)$ at w. Thus, suppose p is non-empty, and let α_i witness $p \Box\!\!\rightarrow q_i$ at w for $i = 1, 2$. If α_1 also witnesses $p \Box\!\!\rightarrow (q_1 \,\&\, q_2)$ at w, we are done. So suppose that α_1 does not witness $p \Box\!\!\rightarrow (q_1 \,\&\, q_2)$. There are two cases now. Either (A) $\alpha_1 \notin q_2$ or (B) $\alpha_1 \in q_2$ but there is a $\beta \in p \setminus (q_1 \cap q_2)$ such that we do not have $\alpha_1 <_w \beta$.

Case (A): Then, since α_2 witnesses $p \Box\!\!\rightarrow q_2$ at w, it follows that $\alpha_2 <_w \alpha_1$. I now claim that it follows that α_2 witnesses $p \Box\!\!\rightarrow (q_1 \,\&\, q_2)$ at w. To see this, choose any $\beta \in p \setminus (q_1 \cap q_2)$. If $\beta \in p \setminus q_1$, then $\alpha_1 <_w \beta$ since α_1 witnesses $p \Box\!\!\rightarrow q_1$ at w, and so $\alpha_2 <_w \beta$ as $\alpha_2 <_w \alpha_1$. If $\beta \in p \setminus q_2$, then also $\alpha_2 <_w \beta$ since α_2 witnesses $p \Box\!\!\rightarrow q_2$ at w. Hence, in either case $\alpha_2 <_w \beta$ and so indeed α_2 witnesses $p \mathrel{\Box\!\!\to} (q_1 \mathbin{\&} q_2)$ at w, and we are done.

Case (B): If $\beta \in p \backslash q_1$, then $\alpha_1 <_w \beta$ since α_1 witnesses $p \mathrel{\Box\!\!\to} q_1$ at w. Thus, in fact, we must have $\beta \in p \backslash q_2$. Since α_2 witnesses $p \mathrel{\Box\!\!\to} q_2$ at w, it follows that $\alpha_2 <_w \beta$. Moreover, either $\beta <_w \alpha_1$ or β and α_1 are w-incommensurable.

I now claim that α_2 witnesses $p \mathrel{\Box\!\!\to} (q_1 \mathbin{\&} q_2)$ at w. For suppose we have a $\gamma \in p \backslash (q_1 \cap q_2)$. Then $\gamma \in p \backslash q_2$ or $\gamma \in p \backslash q_1$ or both. If $\gamma \in p \backslash q_i$ then $\alpha_i <_w \gamma$ since α_i witnesses $p \mathrel{\Box\!\!\to} q_i$. Thus, if $\gamma \in p \backslash q_2$ we have $\alpha_2 <_w \gamma$ and we are done.

Suppose then $\gamma \in p \backslash q_1$. Then, we have $\alpha_1 <_w \gamma$. Now, either $\beta <_w \alpha_1$ or β and α_1 are w-incommensurable. If $\beta <_w \alpha_1$ then since $\alpha_2 <_w \beta$ it follows that $\alpha_2 <_w \gamma$ and we are done. If, on the other hand, β and α_1 are w-incommensurable, it follows that we have $\alpha_2 <_w \beta$, β and α_1 incommensurable, and $\alpha_1 <_w \gamma$, and from (iii) it follows that $\alpha_2 <_w \gamma$, and we are once again done.

Thus, in both cases (A) and (B) we have $p \mathrel{\Box\!\!\to} (q_1 \mathbin{\&} q_2)$ at w.

Thus, (iii) implies (i). Now we will show that (iii) implies (ii). Suppose $p_i \mathrel{\Box\!\!\to} q$ at w is witnessed by α_i for $i = 1, 2$. If α_1 witnesses $(p_1 \vee p_2) \mathrel{\Box\!\!\to} q$ at w, we are done, so suppose that α_1 does not witness $(p_1 \vee p_2) \mathrel{\Box\!\!\to} q$ at w. Thus, there exists a $\beta \in (p_1 \cup p_2) \backslash q$ such that $\alpha_1 \not<_w \beta$. Since α_1 witnesses $p_1 \mathrel{\Box\!\!\to} q$ at w, this is only possible if $\beta \in p_2 \backslash q$. In that case, we have $\alpha_2 <_w \beta$ since α_2 witnesses $p_2 \mathrel{\Box\!\!\to} q$ at w.

I now claim that α_2 witnesses $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} q$ at w. For suppose that $\gamma \in (p_1 \cup p_2)\backslash q$. We must show that $\alpha_2 <_w \gamma$. If $\gamma \in p_2\backslash q$ this follows immediately from the fact that α_2 witnesses $p_2 \mathrel{\Box\!\!\rightarrow} q$ at w. So suppose $\gamma \in p_1\backslash q$. Then, $\alpha_1 <_w \gamma$. Now we have $\alpha_1 \not<_w \beta$. Thus, either α_1 and β are w-incommensurable or $\beta <_w \alpha_1$. In the latter case we have $\alpha_2 <_w \beta <_w \alpha_1 <_w \gamma$ and we are done. In the former case, we have $\alpha_2 <_w \beta$, β and α_1 are w-incommensurable, and $\alpha_1 <_w \gamma$, and it follows from (iii) that $\alpha_2 <_w \gamma$. Hence, indeed, we have $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} q$ at w.

We now need to show that $\sim$(iii) implies $\sim$(i) and $\sim$(ii). Now, if (iii) fails then there is a quintuple of worlds α, β, γ, δ and w such that $\alpha <_w \beta$ and $\gamma <_w \delta$ and β and γ are w-incommensurable, while we do not have $\alpha <_w \delta$.

Now, let $p = \{\alpha, \beta, \gamma, \delta\}$. Let $q_1 = \{\alpha, \beta, \gamma\}$. Let $q_2 = \{\alpha, \gamma, \delta\}$. Observe that $p \mathrel{\Box\!\!\rightarrow} q_1$ at w is witnessed by γ. For if w' is any member of $p\backslash q_1$, then $w' = \delta$ and indeed $\gamma <_w \delta$. Similarly, $p \mathrel{\Box\!\!\rightarrow} q_2$ at w is witnessed by α since the only member of $p\backslash q_2$ is β and $\alpha <_w \beta$.

I now claim that we do *not* have $p \mathrel{\Box\!\!\rightarrow} (q_1 \,\&\, q_2)$ at w. For $q_1 \cap q_2 = \{\alpha, \gamma\}$. Thus, the only possible witnesses to $p \mathrel{\Box\!\!\rightarrow} (q_1 \cap q_2)$ at w are α and γ. But α does not witness $p \mathrel{\Box\!\!\rightarrow} (q_1 \cap q_2)$ at w because $\delta \in p\backslash(q_1 \cap q_2)$ and $\alpha \not<_w \delta$. Neither does γ witness $p \mathrel{\Box\!\!\rightarrow} (q_1 \cap q_2)$ at w because $\beta \in p\backslash(q_1 \cap q_2)$ and $\gamma \not<_w \beta$ since γ and β are w-incommensurable. Hence, indeed, (i) fails.

To see that (ii) also fails, let $p_1 = q_1 = \{\alpha, \beta, \gamma\}$ and let $p_2 = q_2 = \{\alpha, \gamma, \delta\}$. Set $q = \{\alpha, \gamma\}$. Observe that $p_1 \mathrel{\Box\!\!\rightarrow} q$ at w is witnessed by

α since $p_1 \backslash q = \{\beta\}$ and $\alpha <_w \beta$. Also observe that $p_2 \mathrel{\Box\!\!\rightarrow} q$ at w is witnessed by γ since $p_2 \backslash q = \{\delta\}$ and $\gamma <_w \delta$.

On the other hand, we do not have $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} q$ at w. For the only possible witnesses would be the members of q, namely α and γ. But as before, α does not witness $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} q$ at w since $\alpha \not<_w \delta$ and $\delta \in (p_1 \cup p_2) \backslash q$, and γ does not witness it either since $\gamma \not<_w \beta$ and $\beta \in (p_1 \cup p_2) \backslash q$. □

Proof of Theorem 2. First, let us prove (i). Suppose p is non-empty. Then suppose $p \mathrel{\Box\!\!\rightarrow} q_i$ at w is witnessed by α_i for $i = 1, 2$. To obtain a contradiction suppose $p \mathrel{\Box\!\!\rightarrow} {\sim}(q_1 \,\&\, q_2)$ is witnessed by β. Then, $\beta \in p \backslash (q_1 \cap q_2)$. Without loss of generality, suppose $\beta \in p \backslash q_1$. Then, since α_1 witnesses $p \mathrel{\Box\!\!\rightarrow} q_1$ at w, we have $\alpha_1 <_w \beta$. This would contradict the claim that β witnesses $p \mathrel{\Box\!\!\rightarrow} {\sim}(q_1 \,\&\, q_2)$ at w, if we had $\alpha_1 \in q_1 \cap q_2$, so $\alpha_1 \notin q_1 \cap q_2$. As $\alpha_1 \in q_1$, it follows that $\alpha_1 \notin q_2$. Thus, since α_2 witnesses $p \mathrel{\Box\!\!\rightarrow} q_2$ at w, we have $\alpha_2 <_w \alpha_1 <_w \beta$. Since β witnesses $p \mathrel{\Box\!\!\rightarrow} {\sim}(q_1 \,\&\, q_2)$ at w, it follows as before that $\alpha_2 \notin q_1 \cap q_2$, and since $\alpha_2 \in q_2$, it follows that $\alpha_2 \notin q_1$. But then since α_1 witnesses $p \mathrel{\Box\!\!\rightarrow} q_1$ at w, it follows that $\alpha_1 <_w \alpha_2$, which contradicts the claim that $\alpha_2 <_w \alpha_1$. Hence, a contradiction indeed ensues.

Suppose now that $p_1 \cup p_2$ is non-empty and $p_i \mathrel{\Box\!\!\rightarrow} q$ at w is witnessed by α_i for $i = 1, 2$. To obtain a contradiction suppose β witnesses $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} {\sim} q$ at w. Then, $\beta \in (p_1 \cup p_2) \backslash q$. Without loss of generality, suppose $\beta \in p_1 \backslash q$. Then, $\alpha_1 <_w \beta$ since α_1 witnesses $p_1 \mathrel{\Box\!\!\rightarrow} q$ at w. But since β witnesses $(p_1 \vee p_2) \mathrel{\Box\!\!\rightarrow} q$ at w, it follows that $\alpha_1 \notin (p_1 \cup p_2) \cap q$.

However, if α_1 witnesses p_1, then $\alpha_1 \in p_1 \cap q$. Thus, $\alpha_1 \notin p_2$. It follows then, as in the conjunction case, that $\alpha_2 <_w \alpha_1 <_w \beta$. As before, by the witnessing condition on β, it follows that $\alpha_2 \notin (p_1 \cup p_2) \cap q$, and so $\alpha_2 \notin p_1$. By the witnessing condition on α_1 we then have $\alpha_1 <_w \alpha_2$, which contradicts the earlier claim that $\alpha_2 <_w \alpha_1$. $\square$

If w is fixed by the context, we shall write $p^\alpha = \{\beta \in p : \beta \leq_w \alpha\}$, where $\beta \leq_w \alpha$ if and only if either $\beta <_w \alpha$ or $\beta = \alpha$. Observe that $p \mathbin{\Box\!\!\rightarrow}^* q$ at w holds if and only if for every $\alpha \in p \backslash q$ there is a $\beta \in p$ such that $\beta <_w \alpha$ and $p^\beta \subseteq q$. Note also that if α witnesses $p \mathbin{\Box\!\!\rightarrow} q$ at w, then $p^\alpha \subseteq q$.

Proof of Theorem 3. If $p \mathbin{\Box\!\!\rightarrow} q$ at w and p is empty, then (1) is satisfied trivially, and so $p \mathbin{\Box\!\!\rightarrow}^* q$. Suppose p is non-empty. Let β witness $p \mathbin{\Box\!\!\rightarrow} q$ at w. Choose any $\alpha \in p \backslash q$. Since β witnesses $p \mathbin{\Box\!\!\rightarrow} q$ at w, it follows that $\beta <_w \alpha$ and $p^\beta \subseteq q$. Thus, (1) is satisfied, and so $p \mathbin{\Box\!\!\rightarrow}^* q$.

Suppose $p \mathbin{\Box\!\!\rightarrow}^* q$ at w and $p \mathbin{\Box\!\!\rightarrow}^* {\sim} q$ at w. To obtain a contradiction, suppose p is non-empty. Choose $\alpha \in p$. Then either $\alpha \in q$ or $\alpha \in W \backslash q$. Without loss of generality suppose $\alpha \in W \backslash q$ (else replace q with ${\sim} q$). Then, $\alpha \in p \backslash q$. Since $p \mathbin{\Box\!\!\rightarrow}^* q$ at w, there is a $\beta \in p$ such that $\beta <_w \alpha$ and $p^\beta \subseteq q$. Since $p \mathbin{\Box\!\!\rightarrow}^* {\sim} q$ at w, there is then a $\gamma \in p$ such that $\gamma <_w \beta$ and $p^\gamma \subseteq W \backslash q$. Since $\gamma <_w \beta$, we have $\gamma \in p^\beta$, however, and so $\gamma \in q$, which contradicts the facts that $p^\gamma \subseteq W \backslash q$ and $\gamma \in p$. Thus, if p is non-empty, we cannot have both $p \mathbin{\Box\!\!\rightarrow}^* q$ at w and $p \mathbin{\Box\!\!\rightarrow}^* {\sim} q$ at w.

Finally, we need to prove (1) and (2) for $\Box\!\!\to^*$. Suppose that $p \Box\!\!\to^* q_1$ at w and $p \Box\!\!\to^* q_2$ at w.

Fix any $\alpha \in p \backslash (q_1 \cap q_2)$. To show that $p \Box\!\!\to^* (q_1 \,\&\, q_2)$ at w, we need to show that there is a world $\theta \in p$ such that $\theta <_w \alpha$ and $p^\theta \subseteq q_1 \cap q_2$. To do so, proceed as follows. We have $\alpha \in p \backslash q_1$ or $\alpha \in p \backslash q_2$ (or both). Without loss of generality, suppose $\alpha \in p \backslash q_1$. Since $p \Box\!\!\to^* q_1$ at w, there is a $\beta \in p$ such that $p^\beta \subseteq q_1$ and $\beta <_w \alpha$.

Suppose first we have $p^\beta \subseteq q_2$. Then, $p^\beta \subseteq q_1 \cap q_2$, and we are done upon letting $\theta = \beta$.

Next, suppose that p^β is not a subset of q_2. Let $\gamma \in p$ be such that $\gamma \leq_w \beta$ and $\gamma \notin q_2$. Since $p \Box\!\!\to^* q_2$ at w, there is a $\delta \in p$ such that $\delta <_w \gamma$ and $p^\delta \subseteq q_2$. Then $\delta <_w \beta$ and so $p^\delta \subset p^\beta \subseteq q_1$. Thus, $p^\delta \subseteq q_1 \cap q_2$, and so we are done if we let $\theta = \delta$.

It remains to show (2) for $\Box\!\!\to^*$. Suppose $p_1 \Box\!\!\to^* q$ at w and $p_2 \Box\!\!\to^* q$ at w. Fix $\alpha \in (p_1 \cup p_2) \backslash q$. To show that $(p_1 \vee p_2) \Box\!\!\to^* q$, we need to show that there is a $\theta \in p_1 \cup p_2$ such that $\theta <_w \alpha$ and $(p_1 \cup p_2)^\theta \subseteq q$. To do this, suppose without loss of generality that $\alpha \in p_1$.

Then, since $p_1 \Box\!\!\to^* q$ at w, there is a $\beta \in p_1$ such that $\beta <_w \alpha$ and $p_1^\beta \subseteq q$. If $p_2^\beta \subseteq q$, then we are done upon letting $\theta = \beta$. Thus, suppose that p_2^β is not a subset of q. Let $\gamma \in p_2^\beta \backslash q$. Since $p_2 \Box\!\!\to^* q$, there is a $\delta \in p_2$ such that $\delta <_w \gamma$ and $p_2^\delta \subseteq q$. Since $\gamma \leq_w \beta$, it follows that $\delta <_w \beta$. Thus, $(p_1 \cup p_2)^\delta \subseteq p_1^\beta \cup p_2^\delta \subseteq q \cup q = q$. Letting $\theta = \delta$ completes the proof. $\Box$

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