The Measure Problem

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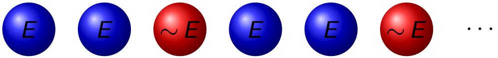
The problem and the claim

- There are infinitely many observers (or types of observers, or observations, or types of observations) in my reference class K—e.g., sharing my subjective state apart from E.
- Infinitely many of them observe E and infinitely many do not, with equal cardinality of the infinities.
- I have information about the arrangement (in spacetime or otherwise) of the members of K and the distribution of observation E in that arrangement.



- I know that I am in K, but have no information that distinguishes me from any other member of K.
- I want to know to what degree I should expect to observe E.
- This is an epistemology question. Physics has told me the arrangement of the members of K and the distribution of E is. Now I want to know *what I should think*—how much confidence I should have in E. Questions that have a "should" in them are not physics questions.
- Claim: In a situation like the above, I cannot make any nontrivial probabilistic assessment of how likely I am to observe E.
- If we allow interval-valued probabilities, I could say $P(\text{I observe } E) = [0, 1]$.
- For simplicity, assume the infinities are all countable.

Radical ramifications

- In infinite multiverse scenarios, and in some infinite universe scenarios, I am in the above situation with respect to all observations.
    - The measure problem is not just about getting *cosmological* predictions.
    - It is also about ordinary day-to-day predictions. I roll a die. What's the probability that it'll be 1?
    - In our scenario, there are infinitely many people with memories like mine who roll a die. Infinitely many of them get 1 and infinitely many get non-1. The measure of those who get 1 is what tells me how likely I am to get 1. (This gets a bit more complicated if we use proxies in the measure.)
    - For a simple cut-off and limit measure, by the Strong Law of Large Numbers almost surely $\lim_{n\to\infty} \frac{|\{i \le n : o_i \text{ rolls } 1\}|}{n} = 1/6$, which is right.
- So, if the measure doesn't work, I cannot make *any* probabilistic assessments.
- So, all empirical science is overthrown.
- So, any scientific argument for such an infinite scenario is self-undermining.
- The setup of the scenario abstracts from the details of the physics, and hence is very general.

Kolmogorov's classical axioms for probability theory

- A probability measure is a function P from a σ-field $\mathcal{F}$ on a sample space Ω to the reals such that
  - K1 $0 \leq P(A)$
  - K2 $P(\Omega) = 1$
  - K3 If $A_1, A_2, \ldots$ are pairwise disjoint ($A_i \cap A_j = \varnothing$ if $i \neq j$), then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$.
- A σ-field $\mathcal{F}$ on Ω is a non-empty set of subsets of Ω closed under complements and countable unions (and hence countable intersections).
- In our scenario, $\Omega = K$, and we want to know the probability of the set $E^* = \{o \in K : o \text{ observes } E\}$.
- Measures based on cut-off and limit methods are defined by $\mu(A) = \lim_{n\to\infty} \frac{|A \cap U_n|}{|U_n|}$, where U_n is some expanding sequence of finite sets whose union is K.
- Let $o_1, o_2, \ldots$ be the observers in K. Then $\mu(\{o_i\}) = 0$. But $\{o_1\} \cup \{o_2\} \cup \cdots = K$. So we violate (K3) if we let $A_i = \{o_i\}$.
- Also, μ is not defined on a σ-field, or even a field (only require closure under complements and finite unions). Easy to find cases where the limits defining $\mu(A)$ and $\mu(B)$ are defined, but the limit defining $\mu(A \cap B)$ is not defined.

Finitely additive measures

- Recall:
  - K1 $0 \leq P(A)$
  - K2 $P(\Omega) = 1$
  - K3 If $A_1, A_2, \ldots$ are pairwise disjoint ($A_i \cap A_j = \varnothing$ if $i \neq j$), then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$.
- Replace (K3) with:
  - K3f If $A_1 \cap A_2 = \varnothing$, then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.
- And define only on a field $\mathcal{F}$. This is a finitely-additive probability.
- The good news:
  - **Theorem** (given Axiom of Choice): There is a finitely-additive probability μ on all subsets of K such that $\mu(A) = \lim_{n\to\infty} \frac{|A \cap U_n|}{|U_n|}$ whenever this limit is defined.
  - **Theorem:** The Central Limit Theorem and the Weak Law of Large Numbers hold for finitely-additive probabilities for bounded independent variables. (And under weaker conditions)
- But while mathematically interesting, finitely-additive probabilities are inadequate for epistemological purposes. And we're doing epistemology here.

Conglomerability, I

- At the heart of the epistemological uses of probability is conditional probability: $P(A|B) = P(A \cap B)/P(B)$.
- Classical probabilities are **conglomerable**: If U is a partition of Ω into disjoint subsets of positive probability, and $P(B|A) \geq \alpha$ for all $A \in U$, then $P(B) \geq \alpha$.
- For a non-conglomerable probability, there is a partition U of Ω into disjoint subsets of positive probability, with $P(B|A) \geq \alpha$ for all $A \in U$ and $P(B) < \alpha$.
- If you perform the experiment, you find out which member A of U you're in. If conglomerability fails here, then you already know that no matter what result you get from the experiment, your probability for B will go up. This is *reasoning to a foregone conclusion*: you know which way the evidence will go *before you get it*. (Kadane, Schervish and Seidenfeld 1986)

Conglomerability, II

Numbers game

You and I each independently and uniformly (well-defined finitely-additive probability) get numbers N_{you} and N_{me} from $1, 2, \ldots$. Person with the bigger number wins. Each sees their own number first. You know that no matter what number n you get, $P(N_{\text{me}} > N_{\text{you}} | N_{\text{you}} = n) = P(N_{\text{you}} > n) = 1$. You will despair!

- Add: standing offer of a side-bet before both numbers are revealed. If $N_{\text{me}} > N_{\text{you}}$, you get \$1; if $N_{\text{you}} > N_{\text{me}}$, you pay me \$1000. Before the numbers are picked, this is crazy. After you see N_{you}, you assign probability 1 to N_{me} being bigger, and so you're rationally compelled to accept the bet. So you should be willing to pay me \$100 not to find out N_{you}.
- Classical probabilities satisfy Good's Theorem (1967): You should never pay not to find out the result of an experiment.
- Non-conglomerable probabilities always fail Good's Theorem (Kadane, Schervish and Seidenfeld 1986).
- **The kicker:** All merely finitely additive probabilities are non-conglomerable (Schervish, Seidenfeld and Kadane 1984), and hence in some situations will result in reasoning to a foregone conclusion and paying not to know.
- Probability functions with such bad properties are epistemically bad.

Conglomerability, III

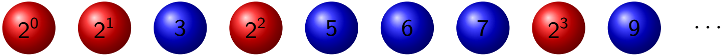
- If I have no evidence distinguishing me from anyone else in K, then for any two observers o_i and o_j in K, the probability that I am o_i and the probability that I am o_j must be the same.
- For suppose I think it's more likely that I am o_i. And suppose you're also a member of K. Then because you're relevantly like me, you'll reason just like me and you'll think it's more likely that *you* are o_i. But that's absurd: We'll be having a disagreement despite having exactly the same evidence.
- (This condition fails for causal-diamond kinds of measures where one is only counting over a finite subset of all the observers.)
- So if P is a decent measure on K, we will have $P(\{o_i\}) = P(\{o_j\})$. But then $1 \geq P(\{o_1, \ldots, o_n\}) = P(\{o_1\}) + \cdots + P(\{o_n\}) = nP(\{o_1\})$ for all n by finite additivity. So $P(\{o_1\})$ must be zero (or infinitesimal, but that won't help). And so our measure is not countably additive because it assigns zero to each $\{o_i\}$ and one to their union.
- So all the decent measures are at best going to be finitely additive. And mere finite additivity leads to non-conglomerability. Which is epistemically problematic. So there are no decent measures.
- It may be that the kinds of cases that lead to the paradoxes are not the ones that are going to come up in the practice of physics. Nonetheless, the fact that the measures lead to such paradoxes shows that they are not the right way to measure epistemic probabilities.

Structure, I

The balls

There are a bunch of red and blue balls on a table. An angel appears and tells you that one of the balls has your name in it. In ten days, you will get a million dollars if the ball with your name inside it is blue. You can't open any of the balls or look in them, but in the meanwhile you can arrange the balls however you like (with the angel's help if you need it). Is there any arrangement that increases the probability of your getting the million? (Cf. Hud Hudson)

- Obviously not!
- But suppose there are infinitely many balls, and you ask the angel to arrange them in a line so that the red ones are at powers of two and the blue ones at all other numbers, reasoning that this increases the probability that the unknown ball is blue.



- Your cleverness is misspent. The order of the balls does not change the probability the ball with your name on it is blue.

Structure, II

Souls

You're one of an infinite number of disembodied souls outside of space and time. God is going to put one into a body every year. God has already decided which soul will have a tough life and which will have an easy life. God says he has two alternative universes he could create. In u_1, the souls with the tough life will be embodied in years numbered with a power of two. In u_2, the souls with the easy life will be embodied in years numbered with a power of two. You prefer a tough life. Which universe do you want God to create?

- It surely doesn't matter.
- Conclusion: The physical structure of the reference class does not affect the probabilities.
- But all the proposed measures are based on the physical structure.
- And have to be. For if we disregard physical structure, our measure will have to be invariant under all permutations. But there is no hope of that, of course. Trivially:
- **Theorem:** There is no permutation-invariant measure on a countably infinite set that is defined on a permutation-invariant field that contains an infinite set with infinite complement.

A fun puzzle

Dice and chests

Smith tosses a die but don't get to see the number. On each of the next six days (numbered $1, 2, \ldots, 6$), Smith will be taken into a room. In that room, there will be a sealed chest that can't be accessed by your or Smith. In that chest there will be Smith's name on a piece of paper if and only if the day number is equal to what her die showed. But there is a twist. There are infinitely many players of the game. Moreover, the organizers will pair them up, so that the players will be taken into their rooms in pairs, such that:

a. The chest contains exactly one name.
b. Nobody is ever paired with anybody twice.

- This can be done, assuming there are infinitely many of each number rolled.
- So, it's day 1. And you observe Smith in the room with Jones. What is the probability that Smith's name is in chest?
- Both $1/2$ and $1/6$ lead to a contradiction.