The Finite

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Descartes on the Finite

I clearly understand that there is more reality in an infinite substance than in a finite one, and hence that my perception of the infinite, i.e. God, is in some way prior to my perception of the finite, i.e. myself. Whenever I know that I doubt something or want something, I understand that I lack something and am therefore not wholly perfect. How could I grasp this unless I had an idea of a more perfect being that enabled me to recognize my own defects by comparison? – Descartes, Third Meditation

- Descartes' notion of the finite is of what falls short of something else.
- In that sense, even mathematically infinite sets are "finite": the set of integers fall short of the set of rationals, which falls short of the set of reals, etc.
- Is there an argument for God from the mathematically finite?

Infinite proofs

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Infinite proofs

- It seems a proof is a sequence of statements each of which is either a premise, an axiom or a consequence of what came before.
- But an infinite proof can prove anything:

  ...
  (−3) Roses stink and roses stink. (By (−4))
  (−2) Roses stink. (By (−3))
  (−1) Roses stink and roses stink. (By (−2))
  (0) Roses stink. (By (−1))
- A proof is a finite sequence of finite statements each of which is either a premise, an axiom or a consequence of what came before.

Numbers

- The standard mathematical characterization of the finite: a set is finite provided that you can number its members $1, 2, ..., n$ for some natural number n.
- But what is a natural number?
- We better not include infinite numbers!
- We can characterize the natural numbers as objects that include a special zero object 0 and a successor operation s (i.e., $sn = n + 1$) that satisfies the Peano Axioms, such as that $sn \neq n$, that every number other than 0 is a successor, etc.
- There turn out to be infinitely many axioms.
- We assume that the Peano Axioms are consistent.
- The "Roses stink" argument can now be ruled out.

Problem: Applying the Peano Axioms

- Let's prove that 2 is a natural number. By definition $2 = ss0$.
  1. 0 is a natural number. (Axiom)
  2. For any natural number n, sn is a natural number. (Axiom)
  3. If 0 is a natural number, $s0$ is a natural number. (By 2)
  4. So, $s0$ is a natural number. (By 1 and 3)
  5. If $s0$ is a natural number, $ss0$ is a natural number. (By 2)
  6. So, $ss0$ is a natural number. (By 4 and 5)
- But to know that this is a proof, we need to know that it has a natural (and hence finite) number of non-axiom steps.
- So we need to know that 4 is a natural number.
- But the analogous proof that 4 is a natural number will take 8 non-axiom steps.
- And we'll need to know that 8 is a natural number.
- Vicious regress!

Reaxiomatizing

- We can add some handy axioms to make the proofs go faster. For instance:
  - $s0$ is a natural number.
  - $ss0$ is a natural number.
  - For any natural number n, $sssn$ is a natural number.
- Now we can prove that n is a natural number in at most $2n/3$ non-axiom steps.
- We've avoided vicious regress.

Non-standard models

- But the problem is that the axioms of arithmetic have non-standard models.
- These are mathematical structures that:
  - satisfy the axioms, but
  - the non-standard naturals include what from our point of view are infinite numbers.
- Given a non-standard model of the naturals, we get non-standard proofs:
  - The steps are numbered with non-standard naturals.
  - The parts of each statement are numbered with non-standard naturals.
- Semantic worry: How do we gain reference to the standard naturals? (Kripkenstein)
- Sceptical worries:
  - How do we know that our naturals are not non-standard from the right point of view?
  - How do we know that what we call "standard proofs" are not actually non-standard?
  - How do we know that we can trust our proofs?

Does it matter?

- If we allow non-standard proofs, we will be able to prove new things.
- Some of these are innocent infinite variations on finite statements, like:

$$0 = 0 \,\&\, 0 = 0 \,\&\, 0 = 0 \,\&\, \cdots \,\&\, 0 = 0$$

  with infinitely many conjuncts.
- But Gödel's Second Incompleteness Theorem shows that there is a non-standard model according to which there is a proof of an inconsistency from the axioms of arithmetic.
- Moreover, there are finite statements that are consistent (no contradiction provable) with respect to standard proof, but inconsistent with respect to some non-standard models of proof.
- This isn't how logic should be.
- And adding more axioms doesn't eliminate all the nasty non-standard models.
- Need some non-axiomatic way to eliminate non-standard models to save the absoluteness of logic.

Physics to the rescue

- Writing down a step of a proof takes a minimum amount of time, due to the speed of light limit.
- An infinite proof would take infinitely long.
- We would never have time to get to the conclusion.
- We can say that the finite is what we can finish counting, say at one item a second.

Problems with physics answer

- Odd that logic should depend on physics.
- Would logic be different if it turned out that there is an end of time?
- Physics could be done in a non-standard model. How do we know we aren't in that boat? That what physicists call a "finite number of seconds" isn't really infinite?
- What explains why our physics is based on the standard model?
- Are we just lucky? That seems irrational to think!
- But if we are lucky, that solves the semantic problem.

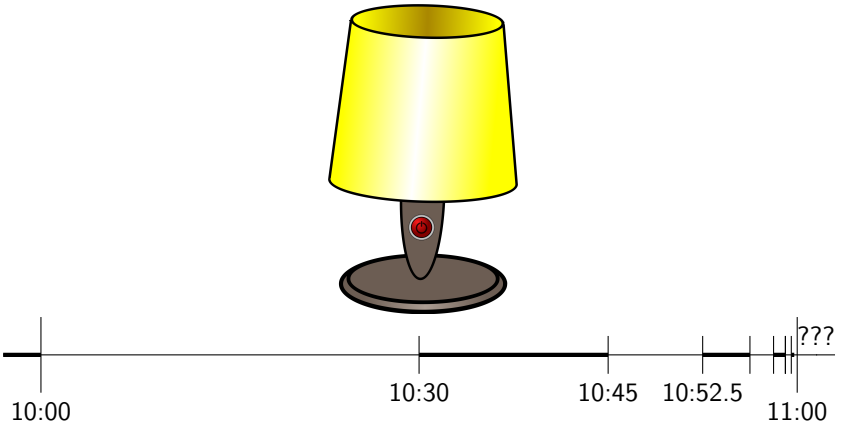
Metaphysical possibility to the rescue

- Perhaps we can use metaphysical necessity to pick out the right models of arithmetic.
- A sentence p is m-contradictory provided that it is metaphysically impossible on any interpretation of its names and predicates in terms of existing objects and instantiable properties.
- A model M of arithmetic is m-acceptable provided that the sentences that are logically contradictory according to M are m-contradictory and *vice versa*.
- A proof is m-acceptable if it is valid according to an m-acceptable M.
- This solves the semantic problem.
- Need an account of metaphysical possibility that does not depend on purely logical possibility. (Note: Some will lead to Cosmological Arguments for God.)
- Explanatory problem: Why do we live in a world where our physical proofs match the m-acceptable ones?
- Sceptical worry: And how do we know we do?

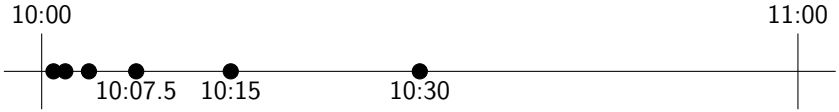
Theism to the rescue

- Theism provides a solution to the problems with the physics solution.
- Mathematics is grounded in ideas in the mind of God (St. Augustine) or in God's power.
- God thus has direct access to all the models, and can choose one that is m-acceptable or minimal or otherwise giving the right notion of proof.
- God can ensure that our words "finite" or "number" match up with that model, either by ensuring we have a human nature with the right semantic properties or by letting us get the meaning of the words by semantic inheritance from God.
- Or God can ensure that the physical world's time sequence and physical abilities for proof-formation match the correct notion of proof.
- God is likely to do this in order that we may have logical knowledge.
- Theism fills out the physics and metaphysical possibility answers.
- Bonus: Explanation of mathematical beauty.

Warmup: Thomson's Lamp



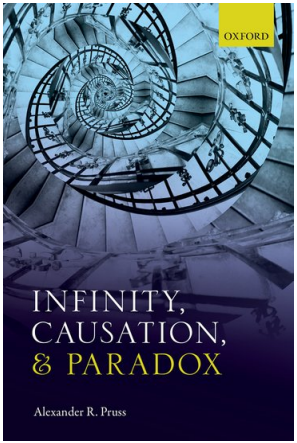
The Grim Reaper Paradox



Shuffling cards

- Suppose I have shuffled an infinite deck of cards numbered $1, 2, 3, \ldots$.
- Two-person game:
  - You and I draw cards from the top of the deck. The person with the biggest number wins.
  - I will be sure that I lost as soon as I see my card.
  - You will be sure that you lost as soon as you saw my card.
  - We'd each be happy to trade!
- I draw 100 cards. After each card I draw, I expect the next one will have a bigger number. But that's stupid!
- To shuffle an infinite deck of cards, use an infinite causal process based on an infinite past:
  - Option 1: On day $-n$, the deck is divided into groups of n cards, each group being simultaneously shuffled.
  - Option 2: Have numbered particles move on line in a random walk with random distances and have their distances today from a fixed center point determine the deck order.

Causal Finitism, I



- Finitism ("there are no infinities") would rule out all such paradoxes.
- But finitism conflicts with mathematics.
- Causal Finitism: Not possible for infinitely many causes to affect a single event.
- Infinitely Reaper observations or observations of dice cannot affect a single event.
- These and many other paradoxes solved!

Causal Finitism, II

- Causal Finitism is a simple principle that rules out many paradoxes.
- But it allows for non-causal infinites, just as mathematics requires.
- We should accept it as true.

Causal Finitism and Counting

- There are genuinely finitely many Fs provided that it is possible for someone think of each of them in a causal sequence, with each thought triggering the next one, and with there being a last thought in the sequence.
- A causal proof: A proof process where one thinks (or utters) each step, but the first, causally because of a previous.
- A genuine proof: A proof that could be realized as a causal proof.
- The metaphysics of causal finitism picks out the true notion of the finite.
- And constrains physical proofs to fit with it, thereby solving the luck problem with the physics approach.

Causal Finitism and Theism

- Causal Finitism underwrites a version of the Kalaam cosmological argument for a first cause:
  1. Something causes something.
  2. There is no infinite regress of causes. (By Causal Finitism)
  3. There is no circularity of causes.
  4. So, there is an uncaused cause.
- Of course, further work is needed to move from an uncaused cause to God. (Aquinas, design arguments, the arguments of the earlier talks.)

Conclusions

- We need an account of the finite for logic to work.
- The axiomatic account fails.
- The physics account suffers from luck and scepticism problems.
- The metaphysical necessity account suffers from luck and scepticism problems.
- The pure theistic account solves the luck and scepticism problems.
  - Bonus: Beauty of mathematics.
- Causal finitism also solves the luck and scepticism problems. And it underwrites the most controversial premise of the Kalaam argument.