REGULAR PROBABILITY COMPARISONS IMPLY THE BANACH-TARSKI PARADOX

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ABSTRACT. Consider the regularity thesis that each possible event has non-zero probability. Hájek challenges this in two ways: (a) there can be nonmeasurable events that have no probability at all and (b) on a large enough sample space, some probabilities will have to be zero. But arguments for the existence of nonmeasurable events depend on the Axiom of Choice (AC). We shall show that the existence of anything like regular probabilities is by itself enough to imply a weak version of AC sufficient to prove the Banach-Tarski Paradox on the decomposition of a ball into two equally sized balls, and hence to show the existence of nonmeasurable events. This provides a powerful argument against unrestricted orthodox Bayesianism that works even without AC. A corollary of our formal result is that if every partial order extends to a total preorder while maintaining strict comparisons, then the Banach-Tarski Paradox holds. This yields an argument that incommensurability cannot be avoided in ambitious versions of decision theory.

1. THE MAIN RESULT

Orthodox Bayesianism holds that rational agents need to (a) assign a probability to *each* event and (b) the probability assigned to contingent events must be strictly between zero and one. For if they assign no probability, or an extreme (zero or one) probability, to an event, then they will be unable to update their credence for the event in a way that is responsive to the evidence: they will be stuck with either no probability[1] or the same extreme probability no matter what evidence may come. Condition (a) we can call "unrestrictedness" and together the two conditions make for "regularity". The "regularity" constraint is very widely accepted (advocates include Shimony 1955; Kemeny 1955; Jeffreys 1961; Stalnaker 1970; Lewis 1980; Appiah 1985).

The main philosophical purpose of this paper is to argue that regularity fails given reasonable approximate symmetry conditions on probabilities in three-dimensional Euclidean setups, regardless of whether the Axiom of Choice (AC) is true. I will do so by giving a theorem that also leads to a second major philosophical conclusion: incommensurability cannot be eliminated from ambitious versions of decision theory that apply to agents who have preferences over large infinite sets of options. But first we need some background.

Alan Hájek (2011) argues that there are two ways that the regularity thesis fails. It fails through nonmeasurability: there are events to which no appropriate probability can be assigned. And it fails due to there being no way to eliminate zero probability events, such as the event of a dart with a perfectly defined tip hitting a particular point.

However, the arguments for nonmeasurable events use versions of the Axiom of Choice (AC), which says that for any set A of non-empty disjoint sets, there is a choice-set C such that C has exactly one element in common with each member of A. Essentially, C

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[1]This observation in the case of no-probability assignment is due to an anonymous referee.

chooses one element from each member of A. Assuming the rest of Zermelo-Fraenkel set theory (ZF) is consistent, AC is known not to be provable from ZF.[2]

Proofs of the existence of nonmeasurable sets come in two varieties. The most common variety in the philosophical literature relies on the existence of Lebesgue-nonmeasurable subsets of the real line, which is proved using a version of AC (see Rudin 1987, 53–54, for a standard proof). These arguments do not have force, however, against a de Finetti style Bayesian who works with merely finitely additive probabilities, since the arguments about Lebesgue measurability assume the countable additivity of probabilities.

The second way to derive the existence of nonmeasurable sets is via the Banach-Tarski Paradox (the most thorough mathematical account of it is in Wagon 1994). The Banach-Tarski Paradox is the claim that a solid three-dimensional ball (i.e., solid spherical region of three-dimensional Euclidean space) can be decomposed into a finite number of disjoint pieces which can be moved by rigid motions (rotations and translations) to generate two solid balls of the same dimensions as the original. It immediately follows that any finitely additive probability measure P that is rigid motion invariant (i.e., $P(A) = P(B)$ if A can be transformed into B via a rigid motion) and defined on a solid cube Ω in three-dimensional Euclidean space must have nonmeasurable subsets (one can also prove this for a ball).

For suppose that P is defined on all subsets of Ω. Let B be a ball in Ω sufficiently small that *two* copies of B can fit in Ω. The Banach-Tarski Paradox (BT) then is the claim that we can write $B = A_1 \cup \cdots \cup A_n \cup C_1 \cup \cdots \cup C_m$ (for some finite n and m[3]), where all the sets here are pairwise disjoint, and where there are rigid motions $\rho_1, \ldots, \rho_n$ and $\tau_1, \ldots, \tau_m$ such that $B_1 = \rho_1 A_1 \cup \cdots \cup \rho_n A_n$ is a disjoint decomposition of one copy of B while $B_2 = \tau_1 C_1 \cup \cdots \cup \tau_m C_m$ is a disjoint decomposition of another copy of B. Given this, by finite additivity and rigid motion invariance:

$$\begin{aligned} P(B) &= \sum_{i=1}^{n} P(A_i) + \sum_{i=1}^{m} P(C_i) \\ &= \sum_{i=1}^{n} P(\rho_i A_i) + \sum_{i=1}^{m} P(\tau_i C_i) \\ &= P(B_1) + P(B_2) = 2P(B). \end{aligned}$$

Subtracting $P(B)$ from both sides, we see that $0 = P(B)$. But this leads to a contradiction. For suppose N is a natural number such that cube Ω_1 whose side-length is $1/N$ times the side-length of the cube Ω fits inside B. Then $P(\Omega_1) = 0$, and so $P(\Omega) \leq N^3 P(\Omega_1) = 0$ since Ω can be covered with N^3 copies of Ω_1. But this violates the total probability axiom $P(\Omega) = 1$.

Note that this is a very general argument. It does not, for instance, assume that the values of P are real numbers. They might be hyperreals. Or they could be members of an ordered commutative group with the total probability axiom replaced by the assumption that $P(\Omega) > 0$.

[2]For an excellent survey of results about AC, see Howard and Rubin (1998).

[3]One can take $n = 3$ and $m = 2$ and the total number of pieces cannot be reduced below 5 (see Theorem 4.7 in Wagon 1994), but we won't take that to be a part of the formal statement of BT.

Like the proof of the existence of Lebesgue non-measurable sets, the proof of the Banach-Tarski (BT) Paradox uses AC. Moreover, if ZF is consistent, it is also consistent with the falsity of BT (Wagon 1994, Th. 13.2).[4]

BT is counterintuitive and some have even taken it as a *reductio* of AC (e.g., Borel 1946, 210). However, we prove that regularity assumptions suffice to commit the regularist to a version of AC sufficiently strong to prove BT. Specifically, we show in ZF, in a very general setting, that the existence of anything like regular probabilities for all subsets of the unit interval $[0, 1]$ implies BT, and BT implies the failure of regularity. So if regularity holds, it fails, and hence regularity fails. Our setting is sufficiently general to subsume both hyperreal-valued probability and conditional probability (Popper function) approaches. The proof adapts Pawlikowski's (1991) proof of BT from the Hahn-Banach Theorem.

Therefore, even without AC, at least one of Hájek's two attacks on regularity succeeds. Either there are no regular probabilities defined for every subset of the unit interval or there are nonmeasurable subsets in any region of three-dimensional Euclidean space big enough to contain a ball. Moreover, as we shall see, switching from regular probabilities to conditional probabilities does not solve the problem.

A corollary of our formal result is that it is true in ZF that if every partial order can be extended to a total preorder while preserving strict inequalities, then BT holds. In particular, this means that decision theorists and economists who think that all partially ordered preference structures can be totalized are committed to a version of AC sufficiently strong for BT, which allows us to make an argument for the possibility of incommensurability in decision theory, independently of AC.

And any assumptions strong enough to prove BT are strong enough to prove the existence of Lebesgue nonmeasurable sets, given that Foreman and Wehrung (1991) have proved in ZF that if all subsets of $[0, 1]$ are Lebesgue measurable, then all subsets of n-dimensional Euclidean space are Lebesgue measurable.[5]

Our main mathematical result is the following theorem of ZF. A *preorder* $\lesssim$ is a reflexive and transitive relation. The preorder is *total* provided that for all x and y, at least one of $x \lesssim y$ and $y \lesssim x$ holds. A total preorder is also known as a *weak order*. We shall write $x < y$ if and only if $x \lesssim y$ and $y \not\lesssim x$, and use $\mathcal{P}\Omega$ to denote the powerset of Ω.

Theorem 1. *Suppose that there is a total preorder $\lesssim$ on the countable members of $\mathcal{P}[0, 1]$ such that: (a) if $A \subseteq B$, then $A \lesssim B$, and (b) if A and B are disjoint sets with B nonempty and $A \lesssim B$, then $A < A \cup B$. Then the Banach-Tarski Paradox holds.*

In the cases we will discuss, $\lesssim$ will be a total preorder on all of $\mathcal{P}[0, 1]$, but in the interests of greatest generality, we give the Theorem in fuller generality.

[4]Solovay (1970) also showed that, on the assumption that ZF is consistent with the existence of an inaccessible cardinal, ZF is consistent with all sets in $\mathbb{R}^n$ being Lebesgue measurable.

[5]See also Easwaran (2014) for further discussion of nonmeasurable sets and the Axiom of Choice. Luxembourg (1973) has shown that the framework of nonstandard analysis used for generating hyperreals can be used to construct Lebesgue nonmeasurable functions, and the existence of nonmeasurable sets follows immediately. Since by the work of Bernstein and Wattenberg (1969) nonstandard analysis can be used to generate regular probabilities for all subsets of $[0, 1]$, and the present paper shows that such regular probabilities imply BT and hence the existence of nonmeasurable sets, the present paper can be seen to provide a generalization of Luxembourg's observation. Not just the existence of hyperreals, but any set of assumptions that implies the existence of regular probabilities for all subsets of $[0, 1]$ with values in a totally ordered field (say, in the field of formal power series considered by Laugwitz 1968 and Pedersen MS) or even a totally ordered commutative group implies the existence of Lebesgue nonmeasurable sets.

We can think of $\lesssim$ as a comparative or "qualitative" probability. Assumption (a) is a significant weakening of standard axioms for qualitative probabilites (e.g., de Finetti, 1931; Villegas, 1967; Suppes, 1994). Assumption (b) is a weakening of Pruss's (2013b) regularity condition that if A is a proper subset of C, then $A < C$. Thus, by itself, Theorem 1 shows that the existence of regular qualitative probabilities on all of $[0, 1]$ is sufficient to generate BT.

We now show how Theorem 1 applies in various scenarios, give a variant that applies to cases of rotational invariance, and along the way discuss philosophical implications. Finally, the proof will be sketched in an appendix. As a referee notes, it may seem surprising that BT follows from non-geometric premises about orderings on subsets of $[0, 1]$. But the surprise should diminish when we consider that BT follows from AC, which is not normally taken to be a geometric claim (interestingly, too, there is a geometric formulation of AC: Bell and Fremlin 1972).

2. Extensions of probability

Hájek (2003) used a counting argument to prove that there is no hope for a classical real-valued probability assignment to be regular—i.e., assign non-zero probability to each non-empty subset—if we have a sample space of uncountable cardinality, say $[0, 1]$.[6] So the interesting cases will be where we are dealing with some extension of the classical notion of probability.

For instance, the conditions of Theorem 1 apply to regular K-valued probabilities for any totally ordered field K, say that of the hyperreals (e.g., Skyrms 1980). Here, we say that P is a K-valued probability on an algebra (i.e., collection closed under finite unions and complements) $\mathscr{F}$ of subsets of Ω provided that we have the Kolmogorov axioms with finite additivity:

(K1) $P(A) \geq 0$
(K2) $P(\Omega) = 1$
(K3) If A and B are disjoint, $P(A \cup B) = P(A) + P(B)$.

And P is regular provided that $P(A) > 0$ for every non-empty A. To see that the Theorem applies, note that if P is a regular K-valued probability on $\mathcal{P}[0, 1]$, then the preorder defined by $A \lesssim B$ if and only if $P(A) \leq P(B)$ clearly satisfies the conditions of Theorem 1. Thus, if we have regular K-valued probabilities on all subsets of $[0, 1]$, we have BT.

Even more generally, we could suppose that K is a commutative group with a compatible total order $\leq$ (the compatibility means that if $a \leq b$, then $a + c \leq b + c$ for any c) and drop K2 (since the element 1 in the group may not be defined), and the same argument would show that the existence of a regular K-valued probability for all subsets of $[0, 1]$ implies BT.

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[6]Hájek's proof uses the assumption that a countable union of finite sets is countable. But while this claim in its full generality uses a version of AC, Hájek only needs the claim that a countable union of finite sets of *real numbers* is countable, which can be proved without AC since the real numbers have a total order. To see this, suppose $A = \bigcup_{i=1}^{\infty} A_i$, where the A_i are finite sets of real numbers. Let $B_1 = A_1$ and $B_n = A_n - \bigcup_{i=1}^{n-1} A_i$ for $n > 1$. Then $A = \bigcup_{i=1}^{\infty} B_i$, and the B_i are disjoint and finite. Let $n_i = |B_i|$ and let $b_{i,k}$, for $1 \leq k \leq n_i$, be the kth smallest member of B_i (where "smallest" is understood relative to the ordinary arithmetical total order on the reals). Then we can enumerate the members of A as follows: $b_{1,1}, \ldots, b_{1,n_1}, b_{2,1}, \ldots, b_{2,n_2}, b_{3,1}, \ldots, b_{3,n_3}, \ldots$. Making this precise uses only ZF.

Now let $\Omega \subseteq \mathbb{R}^3$ be a solid cube. Then Ω has the same cardinality as $[0,1]$ (this can be proved in ZF) and so there is a regular K-valued probability on $\mathcal{P}\Omega$ if and only if there is one on $\mathcal{P}[0,1]$, since a probability on the one space can be transformed into a probability on the other by applying the bijection or its inverse. Thus this follows in ZF:

Corollary 1. *For no commutative totally ordered group K are there K-valued regular rigid-motion invariant probabilities for all subsets of a solid cube.*

For if there are K-valued regular probabilities on $\mathcal{P}\Omega$, then by Theorem 1, BT holds, and hence there aren't any rigid-motion invariant ones.

Corollary 1 is also an easy consequence of the main results of Pruss (2013b), whence this follows more generally for any subset Ω of $\mathbb{R}^n$ containing a circle.

But in our three-dimensional setting, and in the case where K is a totally ordered field, say a hyperreal field, we can prove something stronger than Corollary 1. Any totally-ordered field K contains an isomorphic copy of the rational number field $\mathbb{Q}$. Without loss of generality, we suppose that $\mathbb{Q}$ is a subfield of K. We can then define a standard-value function $\operatorname{St} : K \to \mathbb{R} \cup \{-\infty, \infty\}$ that preserves the arithmetic operations, except where infinities are concerned (just set $\operatorname{St} x = \sup\{y \in \mathbb{Q} : y \le x\}$, where the supremum is the one defined for subsets of $\mathbb{R}$). For any $x \in \mathbb{Q}$, we have $\operatorname{St} x = x$, and if K is actually an extension of $\mathbb{R}$, like the hyperreals are, this is true for any $x \in \mathbb{R}$.

Say that $x \in K$ is infinitesimal provided that $x \neq 0$ and $\operatorname{St} x = 0$. (This holds provided that $0 < |x| < 1/n$ for all natural numbers n.) Write $x \approx y$ provided that $x = y$ or $x - y$ is infinitesimal. This is an equivalence relation. We say that P on $\mathcal{P}\Omega$ is *approximately* rigid motion invariant provided that $P(\rho A) \approx P(A)$ whenever ρ is a rigid motion and A and ρA are subsets of Ω. While by Pruss (2013b) there are no rigid motion invariant K-regular probabilities for all subsets of a circle (or the interval, for that matter), Bernstein and Wattenberg (1969) have shown that there are approximately rigid motion invariant regular hyperreal-valued probabilities on the circle and interval, while Parikh and Parnes (1974) have shown that these can be taken to satisfy the even stronger condition that $P(\rho A)/P(A) \approx 1$ for every non-empty A.[7]

But BT implies that one cannot define *approximately* rigid motion invariant probabilities on a cube Ω. For the argument in Section 1 generalizes to show that if P is such a probability assignment, then $P(B) \approx 2P(B)$. Hence $P(B) \approx 0$. Once again, by covering the cube Ω by copies of a cube small enough to fit inside B, we conclude that $P(\Omega) \approx 0$, which violates the total probability axiom K2. One can also show that the same result holds where Ω is a ball.[8] Thus the following holds in ZF:

Corollary 2. *For no totally ordered field K are there regular K-valued approximately rigid motion invariant probabilities on all subsets of a solid cube or ball Ω.*

This is the first of our two major philosophical upshots.

Corollary 2 is philosophically even more interesting than the result for exact rigid motion invariance. Rigid motion invariance seems to be a reasonable symmetry condition to have on our credences when we want a *uniform* measure on some set in Euclidean space, for instance when we know that a particle or other object with a perfectly defined center

[7]The two conditions coincide where $P(A)$ is non-infinitesimal, but where $P(A)$ and $P(\rho A)$ are both infinitesimal, we automatically have $P(\rho A) \approx P(A)$, while the ratio condition need not be satisfied.

[8]By adding an extra translation, in the setting of our BT paradox we can actually suppose that $\bigcup_{i=1}^m \tau_i C_i$ equals B rather than the disjoint copy B_1 (in fact, normally this is what is proved in proofs of BT). If $B = \Omega$, then we get $P(B) \approx 0$ and an immediate violation of K2.

is in some location of a region but have no further information favoring one place over another. But one might think that our intuitions about symmetry do not cut so finely as to force our credences to be *exactly* rigid motion invariant. Maybe our credences in such cases are only rigid motion invariant up to infinitesimal differences. Or, better, one might think that if we simply have no intuitions about infinitesimal differences, our doxastic state in such symmetric situations should be modeled as not just a single probability function but a family of all the regular approximately invariant hyperreal probabilities (for a recent discussion of family-based approaches to credences, see Joyce 2010).

The above is enough to show that unrestricted Bayesianism—a Bayesianism on which every contingent event has a non-zero probability—fails as a theory of doxastic rationality full stop. For we can imagine cases—like that of the particle with the unknown location—where the only appropriate probability functions are going to be at least approximately rigid motion invariant. For the same reasons, Bayesianism fails as a theory of doxastic rationality for ideal agents, since such agents could be faced with such cases. Unrestricted Bayesianism is ambitiously meant to be applicable in all cases (this could be a stipulation) and so fails.

However, these cases present no difficulty for a modest Bayesianism that makes it a theory of doxastic rationality applicable in a limited though surprisingly broad array of cases or as a toy model for doxastic rationality (see Horwich 1993 for one author with such a limited view of Bayesianism). Thus, the above provides an argument for such modesty in Bayesianism.

The fact that the failures involve nonmeasurable sets does not save the more ambitious forms of Bayesianism. After all, a particle's location being in some nonmeasurable set just is another contingent event, and the ambitious forms of Bayesianism are meant to apply to all contingencies. And few Bayesians will want to completely rule out *a priori* the possibility that nonmeasurable sets are deeply involved in our physics, for instance as in Pitowsky's (1989) interpretation of quantum mechanics centered on nonmeasurable sets.

One might wonder why it matters whether AC is invoked in our corollaries. After all, most mathematicians seem to use AC as a matter of course. And the Hahn-Banach Theorem, which is sufficient for BT (Pawlikowski 1991), is central to functional analysis. Isn't BT, backed up by AC or at least Hahn-Banach, enough to show that ambitious Bayesianism fails?

But Hofweber (MS) has recently argued in response to the anti-regularity arguments of Pruss (2013a) that the choice of the space of values K for a probability measure should be specific to the particular problem one is working on. While Hofweber himself (personal communication) has no difficulties with AC, one might generalize his point. Just as some problems call for a particular hyperreal space of values, our Bayesian could say that some problems call for a particular model of ZF (this, of course, fits best with a structuralist philosophy of mathematics, though a Platonist could also say this). Thus in cases where approximate rigid motion invariance is the right assumption to make, we might work with a model of ZF in which every subset is measurable, like Solovay's model (1970). But Corollary 2 shows that this does not solve the problem—any model of ZF that allows for regular probabilities on a cube or ball ensures that these probabilities are not even approximately invariant.

There is another, more speculative, philosophical implication even if we do not think invariance should be required for probabilities (e.g., Hofweber 2013) and are unbothered by AC. For Bayesianism to be useful to finite agents like us, even somewhat idealized ones, and if credences are to be modeled by a single probability function, then that single probability function needs to be *definable*—there needs to be an explicit finite description of that function, say in ZF+AC, that is uniquely satisfied by it.

Now it seems unlikely that the paradoxical decomposition in the BT Paradox is explicitly definable in ZF+AC. But the methods of proof behind Theorem 1 (including the work we reference from Pawlikowski 1991 and Wagon 1994) are explicitly constructive in that given a total preorder on the subsets of the ball, with the preorder satisfying the conditions of Theorem 1, we can give an explicit construction of a BT decomposition. But likewise one can give an explicit construction of a bijection between $[0, 1]$ and a ball. Consequently, given a preorder on $[0, 1]$ satisfying the requisite conditions, one can explicitly construct from it a BT decomposition. Thus, if one could explicitly construct regular probabilities on $\mathcal{P}[0, 1]$ in ZF+AC (whether these are K-valued probabilities for some commutative group K or just qualitative probabilities), we could also explicitly construct a BT decomposition. Since it is unlikely that a BT decomposition is explicitly definable in ZF+AC, it is also unlikely that an assignment of regular probabilities for all subsets of $[0, 1]$ is definable. Thus, probably, either we should take Bayesianism not to apply to agents like us or we should take the family-of-assignments view. But the argument is speculative, because it *might* turn out that a paradoxical decomposition *is* definable, just as Kanovei and Shelah (2004) have surprisingly shown that there is a definable nonstandard model of the reals.

3. Concentric spheres

One might try to decrease the force of the above arguments against unrestricted Bayesianism by noting that it is difficult to imagine a natural process that produces a point uniformly distributed over a cube or a ball, with a sharp cutoff at the boundary of the cube or ball. Of course, the unnaturalness of the cases should not matter when we are dealing with a general theory of rationality, idealized or not, but the force of the examples against more modest versions will be greater if we have more natural cases to work with. To that end, we consider more rotationally invariant distributions.

The way that BT is actually proved—and this includes the methods of Pawlikowski (1991) and Wagon (1994) that we are using—is that along the way we get a paradoxical decomposition of a sphere (i.e., the surface of a ball). Specifically, it is proved that any sphere S has a partition $A_1, \ldots, A_n, C_1, \ldots, C_m$ and there are rotations ρ_i and τ_i about S's center such that $\rho_1 A_1, \ldots, \rho_n A_n$ partition S and $\tau_1 C_1, \ldots, \tau_n C_n$ also partition S. Here, sets $X_1, \ldots, X_n$ partition a set X provided they are pairwise disjoint and their union is X.

Suppose that a single sphere has such a paradoxical decomposition. Then we can more generally prove a paradoxical decomposition of union of concentric spheres, simply by combining rescaled versions of the paradoxical decomposition of one sphere.[9] Thus the same methods that prove our Theorem 1 also prove:

Theorem 2. *Suppose there is a total preorder on the countable subsets of* Ω *satisfying the conditions of Theorem* 1*. Let* Ω *be a non-empty union of spheres in* $\mathbb{R}^3$ *concentric*

[9] Without loss of generality the center is 0. Let S be one of the concentric spheres in Ω, with radius r. Suppose $A'_1, \ldots, A'_n, C'_1, \ldots, C'_m$ and there are rotations ρ_i and τ_i about S's center such that $\rho_1 A'_1, \ldots, \rho_n A'_n$ partition S and $\tau_1 C'_1, \ldots, \tau_n C'_n$ also partition S. Let $R = \{\alpha \in (0, \infty) : \alpha S \subseteq \Omega\}$, where $\alpha A = \{\alpha a : a \in A\}$. Then let $A_i = \bigcup_{\alpha \in R} \alpha A'_i$ and $C_i = \bigcup_{\alpha \in R} \alpha C'_i$.

about x_0. Thus, there is a partition $A_1, \ldots, A_n, C_1, \ldots, C_m$ and rotations ρ_i and τ_i about x_0 such that $\rho_1 A_1, \ldots, \rho_n A_n$ partition $\Omega - \{0\}$ and $\tau_1 C_1, \ldots, \tau_n C_n$ also partition $\Omega - \{0\}$.

We can define approximate rotational invariance analogously to our definition of approximate rigid motion invariance. The following is then true in ZF:

Corollary 3. *Let Ω be a non-empty region of $\mathbb{R}^3$ invariant under rotation about x_0 and let K be any totally ordered field. Suppose P is a K-valued regular probability on $\mathcal{P}\Omega$ that is approximately invariant under rotations about x_0. Then $P(\{x_0\}) \approx 1$.*

To see this, note that the result is trivial if $\Omega = \{x_0\}$. If $\Omega \neq \{x_0\}$, then $\Omega - \{x_0\}$ is a nonempty union of concentric spheres. The existence of K-regular probabilities on Ω yields a total preorder on $[0, 1]$ satisfying the conditions of Theorem 1, since that preorder can be restricted to the subsets of any one of the spheres in Ω, and each sphere has the same cardinality as $[0, 1]$. Thus, $\Omega - \{x_0\}$ has a paradoxical decomposition as in Theorem 2, so:

$$\begin{aligned} P(\Omega - \{x_0\}) &= \sum_{i=1}^{n} P(A_i) + \sum_{i=1}^{m} P(C_i) \\ &\approx \sum_{i=1}^{n} P(\rho A_i) + \sum_{i=1}^{m} P(\tau C_i) \\ &= P(\Omega - \{x_0\}) + P(\Omega - \{x_0\}) = 2P(\Omega - \{x_0\}). \end{aligned}$$

Hence $P(\Omega - \{x_0\}) \approx 0$, and so $P(\{x_0\}) \approx 1$.

Thus, the only way to get a K-valued regular probability that's approximately invariant is in the trivial case where the probability of being away from x_0 is infinitesimal (in which case approximate invariance comes for free).

The results also generalize to the case of a totally ordered commutative group K. Here, we will need to define $x \approx y$ as holding provided that $n|x - y| \leq |P(\Omega)|$ (where $|a|$ is a if $a \geq 0$ and is $-a$ otherwise) for all natural n. (We leave such generalizations as an exercise to the reader, and we also leave it to the reader to formulate versions for purely qualitative probabilities.)

But now cases of rotational invariance under a single center without any sharp cutoffs at boundaries are natural, in a way in which uniform distribution over a ball or a cube is less so. For instance, imagine a particle (or other object) with a perfectly defined location released at a time t_0 at some point x_0 of Euclidean space and executing a Brownian motion, and for any time $t_1 > t_0$, consider the location of the particle. By the symmetries in Brownian motion, the probability distribution of the location of the particle will be invariant or at least approximately invariant under rotations about x_0, and the probability that the particle will be precisely at x_0 will be at most infinitesimal (infinitesimal if we require regularity). To visualize this, note that the distribution of the position will be a symmetric three-dimensional Gaussian centered on x_0, with standard deviation proportional to $\sqrt{t_1 - t_0}$. So by Corollary 3, there are no K-valued regular probabilities on $\mathcal{P}\mathbb{R}^3$ appropriate to this scenario. And of course, apart from Brownian motions, Gaussian probability distributions are extremely natural and occur with with great frequency in statistical models—they are called "*normal* distributions" after all.

The naturalness of cases like this limits the applicability of Bayesianism.

4. Conditional probabilities

While orthodox unrestricted Bayesianism assumes regular probabilities, a different way to handle problems of zero-probability subsets is instead to take conditional probabilities as primitive (Hájek 2003). Then although there may be zero-probability events, we can still condition on them, and that is what Bayesians really want. We shall show that this does not escape the problems associated with BT.

A full conditional probability (Krauss 1968) on an algebra $\mathscr{F}$ of subsets of Ω is a real valued function P on $\mathscr{F} \times (\mathscr{F} - \{\varnothing\})$ with the properties:

(CP1) for each fixed non-empty B, the function $P(-|B)$ is a finitely additive probability (i.e., satisfies (K1)–(K3) for $K = \mathbb{R}$) with $P(B|B) = 1$,
(CP2) if $A \subseteq B \subseteq C$, then $P(A|B)P(B|C) = P(A|C)$.

This is easily checked to be the same as a Popper function with every non-empty set being normal (see van Fraassen 1976 for definition). And if we want, we can generalize full conditional probabilities to have values in some other ordered field K if we like.

The existence of full conditional probabilities (as well as of hyperreal-valued regular probabilities) on the powerset of arbitrary non-empty set was proved using AC by Krauss (1968). It is easy to see that for any totally ordered field K, the existence of K-valued regular probabilities implies the existence of full conditional probabilities (see Krauss 1968 and McGee 1984). We show that the existence of full conditional probabilities on all subsets of $[0, 1]$ is sufficient for the conditions of the Theorem and hence for BT.

Given full conditional probabilities on $\mathscr{F}$, stipulate $A \lesssim B$ if and only if $P(A|A \cup B) \leq P(B|A \cup B)$.[10] Reflexivity and totality of $\lesssim$ are immediate. Transitivity is a touch more difficult. Suppose $A \lesssim B$ and $B \lesssim C$. Then $P(A|A \cup B)P(A \cup B|A \cup B \cup C) \leq P(B|A \cup B)P(A \cup B|A \cup B \cup C)$, and so by CP2 we have $P(A|A \cup B \cup C) \leq P(B|A \cup B \cup C)$. By the same reasoning $P(B|A \cup B \cup C) \leq P(C|A \cup B \cup C)$. Thus, $P(A|A \cup B \cup C) \leq P(C|A \cup B \cup C)$. If $P(A \cup C|A \cup B \cup C) > 0$, then we can divide both sides of the inequality by $P(A \cup C|A \cup B \cup C)$ and get $P(A|A \cup C) \leq P(C|A \cup C)$ by C2, and hence $A \lesssim C$. Suppose now $P(A \cup C|A \cup B \cup C) = 0$. Then by C1, we have $P(B|A \cup B \cup C) = 1$. Thus, by C2, $P(B|B \cup C)P(B \cup C|A \cup B \cup C) = 1$ and so $P(B|B \cup C) = P(B \cup C|A \cup B \cup C) = 1$. Since $B \lesssim C$, we have $P(C|B \cup C) = 1$ as well, and so $P(C|A \cup B \cup C) = P(C|B \cup C)P(B \cup C|A \cup B \cup C) = 1$. But this contradicts the claim that $P(A \cup C|A \cup B \cup C) = 1$.

We now need only check that $\lesssim$ satisfies the regularity condition (b) in Theorem 1. Suppose A and B are distinct, B is non-empty and $A \lesssim B$. To obtain a contradiction, suppose we do not have $A < A \cup B$. By totality, we have $A \cup B \leq A$. Thus, $P(A \cup B|A \cup (A \cup B)) \leq P(A|A \cup (A \cup B))$. But the left-hand-side equals 1, so the right-hand-side does as well. Thus, $P(A|A \cup B) = 1$. Since A and B are disjoint, it follows from C1 that $P(B|A \cup B) = 0$, which contradicts $A \lesssim B$.

Thus if there are full conditional probabilities on $\mathcal{P}[0, 1]$, then BT holds. Moreover, by Theorem 2 there is a paradoxical decomposition of a union of concentric spheres. Analogously to Corollary 3, we then have in ZF:

[10] A different way of generating a preorder out of conditional probabilities is given by de Finetti (1975): say that $A \lesssim B$ if and only if $P(A - B|A\Delta B) \leq P(B - A|A\Delta B)$, where $A\Delta B$ is the symmetric difference $(A - B) \cup (B - A)$. This ordering has the advantage that if A is a proper subset of B, then $A < B$, but it is somewhat harder to prove transitivity and so this isn't the definition I use here.

Corollary 4. *Let Ω be any region in $\mathbb{R}^3$ invariant under rotation about x_0 and suppose that there are full real-valued conditional probabilities P on $\mathcal{P}\Omega$ that satisfy the invariance condition $P(\rho A|\Omega) = P(A|\Omega)$ for any rotation ρ about x_0. Then $P(\{x_0\}|\Omega) = 1$.*

And if our conditional probabilities take values in some other totally ordered field K, then at least we get $P(\{x_0\}|\Omega) \approx 1$. So we see that there is no way to avoid the problems of nonmeasurable sets, even in rather natural cases like the Brownian motion one, by moving to conditional probabilities and denying AC.

5. Order extension and incommensurability

Next consider the following two claims:

(OE1) Any partial order extends to a total order.

(OE2) Any partial order extends to a total preorder while maintaining strict inequality: if $\leq$ is a partial order on a set X, then there is a total preorder $\leq^*$ on X such that if $x < y$, then $x <^* y$.

In set theory, OE1 is known as the order extension principle. Szpilrajn (1930) proved OE1 using AC, and of course OE2 is an immediate consequence. Felgner and Truss (1999) have shown that, assuming ZF is consistent, OE1 is strictly weaker than the Boolean Prime Ideal Theorem, which in turn is known to be strictly weaker than AC. It is not known if OE2 is strictly weaker than OE1. And it appears to be unknown whether OE1 implies BT. We cannot settle whether OE2 is weaker than OE1, but we can show that OE2, and hence OE1, implies BT.

Corollary 5. *If OE2 holds, then the Banach-Tarski Paradox holds.*

For let $\leq^*$ be an extension of $\subseteq$ on $\mathcal{P}[0,1]$ satisfying the conditions of OE2. Then if A is a proper subset of B, we have $A <^* B$ by OE2, and hence the conditions of Theorem 1 are satisfied.

This shows that whether or not the Banach-Tarski Paradox holds, incommensurability cannot be eliminated from decision theory, or at least an ambitious decision theory.

For if BT holds, then we will have incommensurability in three dimensional situations that are symmetric under rigid motions. Suppose a point is uniformly chosen in some cube Ω, and for any subset C of the cube let G_C be the gamble that pays a dollar if the point is in C and otherwise pays nothing. The following three claims are very reasonable, where $\lesssim$ is a rational agent's non-strict reflexive and transitive preference comparison between gambles and $<$ is the corresponding strict preference ($G_C < G_D$ if and only if $G_C \lesssim G_D$ but not $G_D \lesssim G_C$):

(P1) If C is a subset of Ω and τ is a rigid motion such that τC is a subset of Ω disjoint from C, then it is not the case that $G_C < G_{\tau C}$.

(P2) If C and D are disjoint subsets of Ω, and C' and D' are also disjoint subsets of Ω, with $G_C \lesssim G_{C'}$ and $G_D \lesssim G_{D'}$, then $G_{C\cup D} \lesssim G_{C'\cup D'}$.

(P3) If B_1 and B_2 are two disjoint balls in Ω of the same size, then $G_{B_1} < G_{B_1\cup B_2}$.

The reason for the disjointness condition in P1 is that it is possible to have cases where a set A is a proper subset of a rigid motion τA of itself and in that case one might think there is strict domination (see discussion in Pruss, 2013b).

Given BT and P1–P3, it follows that there are cases of incommensurability, i.e., gambles G_C and G_D such that neither $G_C \lesssim G_D$ nor $G_C \lesssim G_D$. For assume for *reductio* that there is universal commensurability, i.e., that $G_C \lesssim G_D$ or $G_D \lesssim G_C$ for all C and D. Write

$G_C \sim G_D$ provided that both $G_C \lesssim G_D$ and $G_D \lesssim G_C$. Then, P1 plus commensurability implies:

(P1′) If C is a subset of Ω and τ is a rigid motion such that τC is a subset of Ω disjoint from C, then $G_C \sim G_{\tau C}$.

For by commensurability, $G_C \lesssim G_{\tau C}$ or $G_{\tau C} \lesssim G_C$. If both hold, then we have $G_C \sim G_{\tau C}$ as desired. If only one of the two inequalities holds, then it holds strictly, i.e., $G_C < G_{\tau C}$ or $G_{\tau C} < G_C$. The first option violates P1 directly, while the second also violates it when we notice that it would follow that $G_{\tau^{-1}C'} < G_{\tau^{-1}C'}$, where τ^{-1} is the inverse of τ and $C' = \tau C$. Moreover, two applications of P2 yield:

(P2′) If C and D are disjoint subsets of Ω, and C' and D' are also disjoint subsets of Ω, with $G_C \sim G_{C'}$ and $G_D \sim G_{D'}$, then $G_{C\cup D} \sim G_{C'\cup D'}$.

Now, by a slight modification of BT it is possible to decompose a sufficiently small ball B in Ω into two balls B_1 and B_2 of the same size as B that are not only disjoint from each other but that are also disjoint from B (we just have to shift the two balls in the BT decomposition of B in such a way that they don't overlap B, which we can do if the balls are small enough relative to the size of the cube Ω). Specifically, it will be possible to have a disjoint decomposition $B = C_1 \cup \cdots \cup C_m \cup D_1 \cup \cdots \cup D_n$ and rigid motions τ_i and ρ_i such that $\tau_1 C_1 \cup \ldots \tau_m C_m$ is a rigid decomposition of B_1 and $\rho_1 D_1 \cup \ldots \rho_n D_n$ is a rigid decomposition of B_2. Let $C = C_1 \cup \cdots \cup C_m$ and $D = D_1 \cup \cdots \cup D_m$, and note that $B = C \cup D$. Then $G_{C_i} \sim G_{\tau_i C_i}$ and $G_{D_i} \sim G_{\tau_i D_i}$ by P1′, and by repeated application of P2′ we have $G_C \sim G_{B_1}$ and $G_D \sim G_{B_2}$. By P2′ it follows that $G_B = G_{C\cup D} \sim G_{B_1\cup B_2}$. By P1′ we have $G_B \sim G_{B_1}$, and so $G_{B_1} \sim G_{B_1\cup B_2}$, which violates P3.

Thus, given P1–P3, we get incommensurability. Moreover, it's a particularly surprising incommensurability: if we track through the proof, we find that to obtain a contradiction we only needed the commensurability claim that if C' is obtained from C by a rigid motion, then G_C and $G_{C'}$ are commensurable. Thus given BT there will be pairs of sets equivalent under rigid motions gambling on which is incommensurable.[11]

If, on the other hand, BT does not hold, then by Corollary 5 there are partially ordered preference structures that cannot be extended to total preference structures while preserving strict preferences. But non-total preference structures are precisely ones where there are incommensurable options, i.e., A and B such that neither A is at least as preferable as B nor B is at least as preferable as A. Thus, if BT does not hold, incommensurability cannot be eliminated from some preference structures.

Since OE1 (and hence OE2) can be proved without the Axiom of Choice in the case of finite ordered sets, this lack of a totalization will only happen for some infinite preference structures, and one might be willing to bite the bullet and say that we humans only have preferences between finitely many options. This would yield a less ambitious decision theory, one that does not apply to unnatural scenarios far beyond us. But in fact, one preference structure that cannot be totalized without BT seems very natural. Suppose

[11] A similar but simpler argument based on the Hausdorff Paradox rather than the Banach-Tarski Paradox is given in Pruss (forthcoming), but it is not known whether an analogue of our Theorem 1 holds for the version of the Hausdorff Paradox used in that argument. It is also tempting to give an even simpler argument that BT implies incommensurability: BT implies the existence of nonmeasurable sets, and equal gambles on nonmeasurable sets are incommensurable. But the principle that equal gambles on nonmeasurable sets are incommensurable would need refinement. If one set is a proper subset of the other or if the outer measure of one is less than or equal to the inner measure of the other then plausibly there will be a rational preference between the corresponding gambles.

a number X is randomly chosen in $[0,1]$ with every number in $[0,1]$ being possible (the distribution may be, but does not have to be, uniform), and suppose your preferences are between gambles G_A, for $A \subseteq [0,1]$, where the gamble G_A pays a million dollars (or even some particular infinite payoff, if we want to make things more exciting) if X ends up in A, and otherwise pays nothing. It is quite reasonable to prefer G_B to G_A whenever B is a proper superset of A. But Theorem 1 shows that if BT holds, then this preference structure cannot be extended to a total preorder while preserving strict preferability. A decision theory ambitious enough to handle this case thus needs to take account of incommensurability. And typical formal theories assume axioms that rule out incommensurability.

6. Conclusions

The Banach-Tarski Paradox makes it impossible to have rigid-motion invariant probabilities for the outcome of a process that selects a point in a three-dimensional cube, or even probabilities that are invariant up to infinitesimal differences. In situations where the credences should be rigid-motion invariant, at least up to infinitesimal differences, unrestricted Bayesianism fails if the Banach-Tarski Paradox holds. But the Banach-Tarski Paradox can be proved without reliance on the Axiom of Choice from assumptions that the unrestricted Bayesian is apt to accept, such as that there are regular probabilities for all subsets of the cube—perhaps with hyperreal values—or that there are Popper-style conditional probabilities that let us non-trivially condition on each nonempty subset of the cube.[12] Thus, unrestricted Bayesianism fails to account for situations with the kinds of symmetries that obtain here.

Likewise, it can be shown without using the Axiom of Choice that a decision theory ambitious enough to deal with certain infinitary cases must account for incommensurability. Either, the Banach-Tarski Paradox holds, in which case there will be pairs of events that lack probability comparisons, and hence gambles on them will have values incommensurable with one another, or the Banach-Tarski Paradox fails to hold, in which case there will be partial preference structures that cannot be totalized.

Appendix: Sketch of proof of the Theorem

Pawlikowski (1991, Lemma and Note 1) essentially showed how to prove in ZF:

Lemma 1. *Suppose that for every pairwise disjoint collection $\mathscr{B}$ of countable subsets of $\mathbb{R}^3$ there is a function ϕ that assigns to each member $B \in \mathscr{B}$ a collection $\phi(B)$ of subsets of B with the properties:*

(*i*) *If $B \in \mathscr{B}$, $A \in \phi(B)$ and $A \subseteq A' \subseteq B$, then $A' \in \phi(B)$*

(*ii*) *If B_1, B_2, B_3, B_4 are disjoint non-empty subsets of a $B \in \mathscr{B}$, then (a) at most one of the sets B_i is a member of $\phi(B)$ and (b) at least one of the sets $B - B_i$ is a member of $\phi(B)$.*

Then the Banach-Tarski Paradox holds.

A few things need to be noted. First, Pawlikowski in his paper works with a general set X rather than a subset of $\mathbb{R}^3$, but it is clear from the methods of Wagon (1994, Chapters 1–3) that only the case where X is a sphere in $\mathbb{R}^3$ matters for BT. Second, the

[12] I.e., condition in such a way that $P(\varnothing|A) = 0$ if $A \neq \varnothing$. The formal definition of Popper functions (van Fraassen 1974) allows for abnormal sets A such that trivially $P(B|A) = 1$ for all B, even if $B = \varnothing$.

collection corresponding to our $\mathscr{B}$ that Pawlikowski works with is the set of orbits under a free group F on two generators. Since the group F is obviously countable, these orbits are all countable sets.[13] Third, Pawlikowski formulates his proof and remarks in terms of a direct sum of boolean algebras. His work can all be easily reformulated in the setting of our Lemma 1, or else one can apply Pawlikowski Note 1 after letting $\mathbf{B}$ be the boolean direct sum of the boolean algebras $\mathcal{P}B$ as B ranges over the members of $\mathscr{B}$, and letting his set $\mathbf{D}$ be the set of elements of $\mathbf{B}$ of the form $\bigvee_{i=1}^{n}\langle D_i\rangle$ for $D_i \in \bigcup_{B\in\mathscr{B}}\phi(B)$.

Our Theorem 1 follows immediately from Lemma 1 and:

Lemma 2. *Suppose that there is a total preorder $\lesssim$ on the countable members of $\mathcal{P}\mathbb{R}^3$ such that: (a) if $A \subseteq B$, then $A \lesssim B$, and (b) if A and B are disjoint sets with B non-empty and $A \lesssim B$, then $A < A \cup B$. Then the function ϕ defined by $\phi(B) = \{A \subseteq B : B - A < A\}$ satisfies conditions (i) and (ii) of Lemma* 1.

For there is a bijection of $\mathbb{R}^3$ with $[0, 1]$ (this is provable in ZF) and so the existence of the preorder in Lemma 2 follows from the existence of the preorder assumed in the Theorem, and hence BT follows from Lemma 1.

And since the methods of Wagon (1994) that Pawlikowski invokes prove BT by first constructing a paradoxical decomposition of the sphere, we also get Theorem 2.

It remains to prove Lemma 2. Write $X \sim Y$ whenever $X \lesssim Y$ and $Y \lesssim X$.

Proof of Lemma 2. Suppose $A \subseteq A' \subseteq B$ and $A \in \phi(B)$. Then $A \lesssim A'$. Moreover, we have $B - A' \subseteq B - A$ and so $B - A' \lesssim B - A$. Thus, $B - A' \lesssim B - A < A \lesssim A'$ and so $A' \in \phi(B)$. We thus have (i) from Lemma 1.

Observe that for each $A \subseteq B$, at most one of A and $B - A$ is in $\phi(B)$.

Now suppose that B_1, B_2, B_3, B_4 are disjoint non-empty subsets of B. Suppose first that $B_i \in \phi(B)$ and $j \neq i$. Then $B_i \subseteq B - B_j$ by disjointness and so $B - B_j \in \phi(B)$, and hence $B_j \notin \phi(B)$. This yields (ii)(a) in Lemma 1.

It remains to show (ii)(b) in Lemma 1. Suppose i is such that $B_i \not\lesssim B - B_i$. By totality, we have $B - B_i < B_i$ and so $B_i \in \phi(B)$. We know this can happen for at most one of the i. Thus, there are at least three distinct indices i such that $B_i \lesssim B - B_i$. Let I be a set of three such indices. If any one of the $B_i \lesssim B - B_i$ inequalities is strict, we have (ii)(b).

So suppose that none of the inequalities are strict. Thus, $B_i \sim B - B_i$ for $i \in I$. Fix distinct i and j in I. Then $B_i \sim B - B_i$ and $B_j \sim B - B_j$. By disjointness $B_i \subseteq B - B_j$, so $B_i \lesssim B - B_j \lesssim B_j$. By the same token $B_j \lesssim B_i$, and so $B_j \sim B_i$. But i and j were arbitrary distinct indices in I. Thus, if now we write $I = \{i, j, k\}$, we have $B_i \sim B_j \sim B_k$. But by property (b) of our preorder, we have $B_i < B_i \cup B_j$, since B_j is non-empty. Since $B_i \cup B_j \subseteq B - B_k$ by disjointness, we have $B_k \sim B_i < B_i \cup B_j \lesssim B - B_k$ and it follows that $B_k < B - B_k$, contrary to the assumption that none of the three inequalities were strict.[14] $\square$

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