FAILURES OF STRICT PROPRIETY IN PROPER
SCORING RULES

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Let \( \Omega \) be finite.

For \( v \in [0, 1]^{\Omega} \) and \( w \in [-\infty, \infty)^{\Omega} \), write
\[
\langle v, w \rangle = \sum_{\omega \in \Omega} v(\omega)w(\omega).
\]

Let \( P = \{ p \in [0, 1]^\Omega : \sum_{\omega} p(\omega) = 1 \} \).

**Proposition 1.** Let \( s : P \to [-\infty, \infty)^\Omega \) be such that \( \langle p, s(p) \rangle \) is always finite and \( s \) is continuous. Suppose we have propriety: \( \langle p, s(q) \rangle \leq \langle p, s(p) \rangle \) for all \( p, q \in P \). Then if equality holds for some \( p \) and \( q \) in \( P \), we must have \( s(p) = s(q) \).

**Lemma 1.** Under the conditions of the Proposition, if we have equality for some \( p \) and \( q \), then \( \langle r, s(q) \rangle \leq \langle r, s(p) \rangle \) for all \( r \in P \).

**Proof of Lemma 1.** Let \( p_\varepsilon = (1 - \varepsilon)p + \varepsilon r \) for \( \varepsilon \in [0, 1] \). Then:
\[
(1 - \varepsilon)\langle p, s(p) \rangle + \varepsilon\langle r, s(q) \rangle = (1 - \varepsilon)\langle p, s(q) \rangle + \varepsilon\langle r, s(q) \rangle = \langle p_\varepsilon, s(q) \rangle \\
\leq \langle p_\varepsilon, s(p_\varepsilon) \rangle \\
= (1 - \varepsilon)\langle p, s(p_\varepsilon) \rangle + \varepsilon\langle r, s(p_\varepsilon) \rangle \\
\leq (1 - \varepsilon)\langle p, s(p) \rangle + \varepsilon\langle r, s(p) \rangle.
\]

using the assumed equality in the propriety inequality, and applying the propriety inequality twice. Since \( \langle p, s(p) \rangle \) is finite, for \( \varepsilon > 0 \) we must have
\[
\langle r, s(q) \rangle \leq \langle r, s(p) \rangle.
\]

But \( \langle r, s(p_\varepsilon) \rangle \to \langle r, s(p) \rangle \) as \( \varepsilon \to 0^+ \) by continuity.

**Proof of Proposition 1.** Applying Lemma 1, we have \( \langle r, s(q) \rangle \leq \langle r, s(p) \rangle \) for all \( r \in P \). In particular \( \langle q, s(q) \rangle \leq \langle q, s(p) \rangle \). By propriety, we must have equality. Applying Lemma 1 again, we have \( \langle r, s(p) \rangle \leq \langle r, s(q) \rangle \) for all \( r \in P \). Hence, \( \langle r, s(p) \rangle = \langle r, s(q) \rangle \) for all \( r \in P \), and thus \( s(p) = s(q) \).

**References**