FAILURES OF STRICT PROPRIETY IN PROPER SCORING RULES

ALEXANDER R. PRUSS

Let Ω be finite.

For $v \in [0,1]^\Omega$ and $w \in [-\infty,\infty)^\Omega$, write

$$\langle v, w \rangle = \sum_{\substack{\omega \in \Omega \\ v(\omega) \neq 0}} v(\omega) w(\omega).$$

Let $\mathcal{P} = \{p \in [0,1]^\Omega : \sum_\omega p(\omega) = 1\}$.

Proposition 1. *Let $s : \mathcal{P} \to [-\infty,\infty)^\Omega$ be such that $\langle p, s(p)\rangle$ is always finite and s is continuous. Suppose we have propriety: $\langle p, s(q)\rangle \le \langle p, s(p)\rangle$ for all $p, q \in \mathcal{P}$. Then if equality holds for some p and q in $\mathcal{P}$, we must have $s(p) = s(q)$.*

Lemma 1. *Under the conditions of the Proposition, if we have equality for some p and q, then $\langle r, s(q)\rangle \le \langle r, s(p)\rangle$ for all $r \in \mathcal{P}$.*

Proof of Lemma 1. Let $p_\varepsilon = (1-\varepsilon)p + \varepsilon r$ for $\varepsilon \in [0,1]$. Then:

$$\begin{aligned}
(1-\varepsilon)\langle p, s(p)\rangle + \varepsilon\langle r, s(q)\rangle &= (1-\varepsilon)\langle p, s(q)\rangle + \varepsilon\langle r, s(q)\rangle \\
&= \langle p_\varepsilon, s(q)\rangle \\
&\le \langle p_\varepsilon, s(p_\varepsilon)\rangle \\
&= (1-\varepsilon)\langle p, s(p_\varepsilon)\rangle + \varepsilon\langle r, s(p_\varepsilon)\rangle \\
&\le (1-\varepsilon)\langle p, s(p)\rangle + \varepsilon\langle r, s(p_\varepsilon)\rangle.
\end{aligned}$$

using the assumed equality in the propriety inequality, and applying the propriety inequality twice. Since $\langle p, s(p)\rangle$ is finite, for $\varepsilon > 0$ we must have

$$\langle r, s(q)\rangle \le \langle r, s(p_\varepsilon)\rangle.$$

But $\langle r, s(p_\varepsilon)\rangle \to \langle r, s(p)\rangle$ as $\varepsilon \to 0+$ by continuity. □

Proof of Proposition 1. Applying Lemma 1, we have $\langle r, s(q)\rangle \le \langle r, s(p)\rangle$ for all $r \in \mathcal{P}$. In particular $\langle q, s(q)\rangle \le \langle q, s(p)\rangle$. By propriety, we must have equality. Applying Lemma 1 again, we have $\langle r, s(p)\rangle \le \langle r, s(q)\rangle$ for all $r \in \mathcal{P}$. Hence, $\langle r, s(p)\rangle = \langle r, s(q)\rangle$ for all $r \in \mathcal{P}$, and thus $s(p) = s(q)$. □

References

[1] Joel B. Predd, Robert Seiringer, Elliott H. Lieb, Daniel N. Osherson, H. Vincent Poor, and Sanjeev R. Kulkarni. 2009. "Probabilistic Coherence and Proper Scoring Rules", *IEEE Transactions on Information Theory* 55:4786–4792.